Higher Idempotent Completion

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Introduction

The concept of idempotent completeness and idempotent completion has existed in some form or another at least since the early 1970s. Idempotents themselves have been studied for far longer than that of course, for example in the form of projections in linear algebra or idempotent elements in ring theory. One reason for the interest in idempotents is that they often appear in many of the most fundamental constructions, one example being direct sums.

A lot of the time, we are interested in whether a given idempotent is split, approximately meaning whether it can be written in terms of a subobject on which it acts as the identity and onto which everything else gets projected, as is the case for projections in linear algebra. A category in which every idempotent splits gets to be called idempotent complete. In cases where a given idempotent does not necessarily split, we are limited in the constructions we can perform. So it is of no surprise that mathematicians have sought to find ways to alter categories by adding objects and morphisms such that any given idempotent splits in this bigger category, while only adding a minimal amount of new objects and morphisms. It further is of no surprise that this construction which explicitly seeks to split idempotents was first described not in an abstract category theory paper but by Max Karoubi in theorem 6.10 of his introduction to K-theory [Kar78]. This construction, called idempotent completion, also carries the name Karoubi completion in honour of his work.

Another development in the early 1970s was an interest in so-called absolute colimits, colimits which are preserved by every functor. This was done in the context of $\mathcal{V}$-enriched categories, i.e., categories where each Hom-set carries the structure of an object in $\mathcal{V}$ for an appropriate category $\mathcal{V}$. In the case of $\mathbf{Ab}$-enriched categories, finite direct sums are preserved by every $\mathbf{Ab}$-enriched functor. So once again, these absolute colimits classify some fundamental constructions and mathematicians are interested in adding these colimits to a category whenever they don't exist automatically. One way of doing this was described by F. William Lawvere in [Law73] in the context of $\mathcal{V}$-enriched categories. He calls this construction Cauchy completion since, among many other things, it generalises the notion of the Cauchy completion of a metric space. Furthermore, he already notes what we will see later in this thesis, namely, that an ordinary $\mathbf{Set}$-enriched category is Cauchy complete if and only if every idempotent in it splits. This relationship between idempotents and absolute colimits has been studied further in the 1970s and 1980s as described in the survey article [BD86].

Although the notion of higher categories in the form of bicategories has existed since the 1960s, there seemingly was no interest in studying higher categorical analogues of idempotents and idempotent splittings until relatively recently, sparked by applications. When before one was interested in constructions such as direct sums in the category of $\Bbbk$-vector spaces, mathematicians are now interested in direct sums in higher categorical analogues of the category of vector spaces. These types of higher categories have risen to prominence since they supply the algebraic data needed to define extended topological quantum field theories and similar constructions. Two papers which, motivated by these applications, have begun to sketch definitions of higher idempotents and higher idempotent completions are [DR18] by Christopher J. Douglas and David J. Reutter and [GJF19] by Davide Gaiotto and Theo Johnson-Freyd. The latter tackles the problem in a more general setting.

In the first section of this thesis, we will revisit 1-categorical Karoubi and Cauchy completion in such a way that it makes the higher categorical generalisation seem most apparent. The following four theorems are the main results of this section and can be readily found in the literature.

Theorem 1 (see propositions 1.9, 1.10 and corollary 1.13)**.** *The Karoubi completion of a category is its idempotent completion.*

By this, we mean that it is a completion under splitting idempotents and it is in some sense minimal among all such possible completions. We also prove the following statement about Karoubi completion, which is also well known but less documented.

Theorem 2 (see theorem 1.12)**.** *Karoubi completion defines a left adjoint 2-functor* $\widehat{(-)}\colon \mathsf{Cat} \to \mathsf{Cat}_{\mathrm{ic}}$ *from the 2-category of categories into the 2-category of idempotent complete categories.*

This will be of much use later in the thesis. For the Cauchy completion of a category, we get the following analogous statement.

Theorem 3 (see propositions 1.23, 1.26 and corollary 1.29)**.** *The Cauchy completion of a category is its completion under absolute colimits.*

This leads us to the following statement about idempotent completeness and completeness under absolute colimits.

Theorem 4 (see corollary 1.28)**.** *A category is idempotent complete if and only if it is complete under absolute colimits.*

This is a consequence of theorem 1.27 which tells us that the Karoubi completion and the Cauchy completion of a given category are equivalent.

In the second section of this thesis, we will lay down the necessary bicategorical prerequisites for this thesis and prove the following statement along the way.

Construction 5 (see construction 2.44 and theorem 2.45). One can explicitly construct weighted colimits in the 2-category **Cat**.

A weighted colimit is a bicategorical version of a colimit in an ordinary category. This construction is analogous to that of colimits in the category **Set**, but has not been done anywhere this explicitly.

In the third section, we will introduce the definitions of 2-idempotents and their splittings and of the Karoubi completion of a locally idempotent complete bicategory following [GJF19], locally idempotent complete meaning that every Hom-category is idempotent complete. We will then show the following bicategorical analogue of theorem 1.

Theorem 6 (see propositions 3.7, 3.8 and theorem 3.10). *The Karoubi completion of a locally idempotent bicategory is its idempotent completion.*

In the last section of this thesis, we go on to define the Cauchy completion of a bicategory analogously to its 1-categorical counterpart and get the following statement as a bicategorical counterpart to theorem 3.

Theorem 7 (see propositions 4.8, 4.10 and corollary 4.13). *The Cauchy completion of a locally idempotent bicategory is its completion under absolute weighted colimits.*

Finally, we have the following bicategorical analogue of theorem 4.

Theorem 8 (see corollary 4.12). *A locally idempotent complete bicategory is idempotent complete if and only if it is complete under absolute weighted colimits.*

This is a consequence of theorem 4.11, which tells us that the Karoubi completion and the Cauchy completion of a given locally idempotent bicategory are equivalent. So far, the proof of this theorem has only been sketched in [GJF19].

We require the reader of this thesis to be familiar with ordinary category theory. To avoid set-theoretic issues, we assume that all our categories and colimits are small for an appropriate Grothendieck universe.

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Contents

1 The 1-Categorical Case

1.1 Idempotents and their Splittings

Definition 1.1. (Idempotent) An *idempotent* in a category $\mathcal{C}$ consists of an endomorphism $p:A \to A$ on some object A in $\mathcal{C}$ with the property that $p^2 = p$.

Example 1.2. In linear algebra these morphisms are often called projections since a linear idempotent $p:V \to V$ projects each element of V into the subspace $\mathrm{im}(p)$ and acts as the identity on $\mathrm{im}(p)$. This can be expressed by the fact that the corestriction $p|^{\mathrm{im}(p)}:V \to \mathrm{im}(p)$ and the inclusion $\iota:\mathrm{im}(p) \to V$ satisfy $p|^{\mathrm{im}(p)} \circ \iota = \mathrm{id}_{\mathrm{im}(p)}$. They also satisfy $\iota \circ p|^{\mathrm{im}(p)} = p$.

Definition 1.3. (Split Idempotent) In general, whenever we have an idempotent $p:A \to A$ in a given category $\mathcal{C}$ for which we have a second object B and morphisms $f:A \to B$ and $g:B \to A$ such that $f \circ g = \mathrm{id}_B$ and $g \circ f = p$, we say that the idempotent p *splits*. We say that B is a *splitting* of p and B is a *retract* of A.

Definition 1.4. (Idempotent Complete Category) When, for a given category $\mathcal{C}$, every idempotent in $\mathcal{C}$ splits, we call $\mathcal{C}$ *idempotent complete*.

A splitting of an idempotent can also be given by a colimit.

Proposition 1.5. *Let $\mathcal{C}$ be a category, A an object in $\mathcal{C}$, and $p:A \to A$ an idempotent. The coequaliser of*

$$A \underset{p}{\overset{\mathrm{id}_A}{\rightrightarrows}} A$$

defines a splitting of p. Furthermore every splitting of p defines such a coequaliser.

Proof. A colimit of the above diagram consists of an object B and a morphism $f:A \to B$ such that $f \circ p = f \circ \mathrm{id}_A = f$ which satisfies the universal property of the colimit. Since $p:A \to A$ also has the property that $p \circ p = p \circ \mathrm{id}_A$, there exists a morphism $g:B \to A$ such that $g \circ f = p$. Furthermore we have $f \circ g \circ f = f \circ p = f = \mathrm{id}_B \circ f$ and since f is colimiting, $f \circ g = \mathrm{id}_B$. Thus the coequaliser defines a splitting of p.

Now let B be an object with morphism $f:A \to B$ and $g:B \to A$ such that $f \circ g = \mathrm{id}_B$ and $g \circ f = p$ and let C be another object with a morphism $h:A \to C$ such that $h \circ p = h$. We have a morphism $h \circ g:B \to C$ which satisfies $h \circ g \circ f = h \circ p = h$. Now let $k:B \to C$ be another morphism such that $k \circ f = h$, we then have $h \circ g = k \circ f \circ g = k \circ \mathrm{id}_B = k$. Thus $f:A \to B$ defines a coequliser of id_A and p. □

Corollary 1.6. *A splitting of an idempotent is unique up to unique isomorphism.*

Definition 1.7. (Free Walking Idempotent) We define $\clubsuit_1$ to be the category with one object X and one non-identity morphism $p:X\to X$ which satisfies $p^2=p$. We call this category the *free walking idempotent*. It carries this name since the data of an idempotent in a given category $\mathcal{C}$ is equivalent to that of a functor $F:\clubsuit_1\to\mathcal{C}$. Furthermore the coequaliser of proposition 1.5 can then be expressed as the colimit of the corresponding functor.

We already note here that the category $\clubsuit_1$ is special. A splitting of an idempotent is defined purely equationally. Since functors preserve equations, functors preserve splittings, i.e., if an idempotent $p:A\to A$ splits via $f:A\to B$ and $g:B\to A$ in $\mathcal{C}$ and $F:\mathcal{C}\to\mathcal{D}$ is a functor, the idempotent $Fp:FA\to FA$ splits via $Ff:FA\to FB$ and $Fg:FB\to FA$ in $\mathcal{D}$. Since an idempotent splitting is defined by a colimit of a functor out of $\clubsuit_1$, $\clubsuit_1$ has the property that every colimit of a functor out of it gets preserved by every functor with appropriate domain.

We have just now already seen that the category $\mathbf{Vect}_{\mathbb{k}}$ of $\mathbb{k}$-vector spaces and linear maps is indeed idempotent complete. But very much not all categories are. One example is the category **Rel** of sets and relations. The relation $\{(0,0),(0,1)\}$ on the set $\{0,1\}$ defines an idempotent and one can show that this idempotent does not split in Rel. In the following section we will see how we can construct an idempotent complete category from a given category. This construction will turn out to be minimal in the sense that it does not add more than is required.

1.2 Karoubi Completion

Definition 1.8. (Karoubi completion) Let $\mathcal{C}$ be a category. We define the *Karoubi completion* $\widehat{\mathcal{C}}$ to be the category with the following data.

- Objects in $\widehat{\mathcal{C}}$ are idempotents in $\mathcal{C}$, i.e., an object in $\widehat{\mathcal{C}}$ consists of an object A in $\mathcal{C}$ and a morphism $p:A\to A$ in $\mathcal{C}$ such that $p^2=p$. We will denote this object by A_p.
- A 1-morphism between A_p and B_q is given by a morphism $f:A\to B$ in $\mathcal{C}$ such that $f\circ p=f$ and $q\circ f=f$.
- Composition of morphisms in $\widehat{\mathcal{C}}$ is given by the composition in $\mathcal{C}$.
- The identity morphism on A_p is given by $p:A_p\to A_p$.

The morphism in this category are also sometimes called bilodules since they are in some sense decategorified bimodules and hence one letter away from bimodules. The action of a ring on an abelian group is replaced by the action of an idempotent on a morphism, this action being equality.

We first will show that $\widehat{\mathcal{C}}$ is a completion of $\mathcal{C}$ in the sense that $\mathcal{C}$ embeds into $\widehat{\mathcal{C}}$ and is equivalent to it if $\mathcal{C}$ was already idempotent complete.

Proposition 1.9. *For every category $\mathcal{C}$, there exists a fully faithful functor $\iota_{\mathcal{C}}:\mathcal{C}\to\widehat{\mathcal{C}}$. If $\mathcal{C}$ is furthermore idempotent complete, i.e., every idempotent splits in $\mathcal{C}$, this functor is an equivalence.*

Proof. We define $\iota_{\mathcal{C}}$ to map an object A in $\mathcal{C}$ onto the object A_{id_A} in $\widehat{\mathcal{C}}$ and a morphism $f:A\to B$ onto $f:A_{id_A}\to B_{id_B}$. Since any morphism $f:A\to B$ has the property that $f\circ id_A=f$ and $id_B\circ f=f$, we see that $\iota_{\mathcal{C}}$ is fully faithful.

Now assume that $\mathcal{C}$ is idempotent complete and let A_p be an object in $\mathcal{C}$. We know that the idempotent $p:A\to A$ splits in $\mathcal{C}$, which means there exists an object B in $\mathcal{C}$ and morphisms $f:A\to B$ and $g:B\to A$ such that $g\circ f=p$ and $f\circ g=id_B$. These define morphisms $f:A_p\to B_{id_B}$ and $g:B_{id_B}\to A_p$ since $f\circ p=f\circ g\circ f=id_B\circ f=f$ and $p\circ g=g\circ f\circ g=g\circ id_B=g$.

We now have $g\circ f=p=id_{A_p}$ and $f\circ g=id_B=id_{B_{id_B}}$. Thus it follows that $A_p\cong B_{id_B}$ and $\iota_{\mathcal{C}}$ is an equivalence of categories. □

For $\widehat{\mathcal{C}}$ to be the completion of $\mathcal{C}$ under splitting idempotents, we want $\widehat{\mathcal{C}}$ itself to be idempotent complete, which we will show in the following.

Proposition 1.10. *For any category $\mathcal{C}$, the Karoubi completion $\widehat{\mathcal{C}}$ is idempotent complete.*

Proof. Let A_p be an object in $\widehat{\mathcal{C}}$ and $e:A_p\to A_p$ be an idempotent on A_p, i.e., $e\circ p=e$, $p\circ e=e$ and $e^2=e$. It therefore follows that A_e is an object in $\widehat{\mathcal{C}}$ and we have the morphisms $e:A_p\to A_e$ and $e:A_e\to A_p$. Composing these morphisms, we see that $e\circ e=e:A_p\to A_p$ and $e\circ e=e=id_{A_e}:A_e\to A_e$. Thus the idempotent $e:A_p\to A_p$ splits. □

Since an idempotent $p:A\to A$ in $\mathcal{C}$ defines an idempotent $p:A_{id_A}\to A_{id_A}$ in $\widehat{\mathcal{C}}$, we also get the following statement.

Remark 1.11. For an object A_p in $\widehat{\mathcal{C}}$, A_p is a splitting of the idempotent p on A_{id_A}.

Finally, we want to show that the Karoubi completion $\widehat{\mathcal{C}}$ is universal among all possible idempotent completions of $\mathcal{C}$, which is why we can call it *the* idempotent completion of $\mathcal{C}$. By universal, we mean that for any functor F from $\mathcal{C}$ into an arbitrary idempotent complete category $\mathcal{D}$, there is a functor $F' : \widehat{\mathcal{C}} \to \mathcal{D}$ such that $F' \circ \iota_{\mathcal{C}} \cong F$.

For this we will invoke the language of 2-categories, i.e., categories where each Hom-set carries the structure of a category such that composition of morphisms is functorial. A 2-functor between 2-categories is then a functor together with functors for each Hom-category. Later in this thesis, we will properly introduce 2-categories and their weakened variants, bicategories, but for now we will continue since the reader needs only be familiar with the 2-category **Cat** of categories, functors and natural transformations.

This universality will take the form of an adjunction $\widehat{(-)} \dashv \mathcal{U}$ where $\widehat{(-)} : \mathbf{Cat} \to \mathbf{Cat}_{\mathrm{ic}}$ is Karoubi completion and $\mathcal{U} : \mathbf{Cat}_{\mathrm{ic}} \to \mathbf{Cat}$ is the forgetful functor which forgets, that an idempotent complete category is idempotent complete, where we define $\mathbf{Cat}_{\mathrm{ic}}$ to be the full subcategory of idempotent complete categories of the 2-category **Cat**.

Theorem 1.12. *The Karoubi completion defines a 2-functor* $\widehat{(-)} : \mathsf{Cat} \to \mathsf{Cat}_{\mathrm{ic}}$ *which is left adjoint to the forgetful 2-functor* $\mathcal{U} : \mathsf{Cat}_{\mathrm{ic}} \to \mathsf{Cat}$.

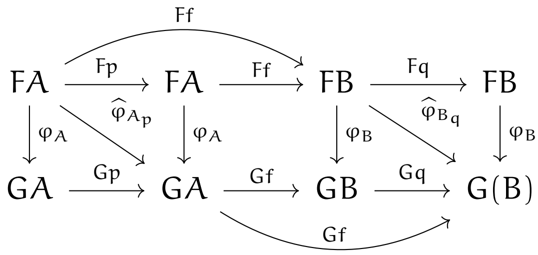
Proof. The 2-functor $\widehat{(-)}$ maps a category $\mathcal{C}$ to the category $\widehat{\mathcal{C}}$. A functor $F : \mathcal{C} \to \mathcal{D}$ is mapped onto the functor $\widehat{F} : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ which maps an idempotent A_p onto the idempotent FA_{Fp} and a morphism $f : A_p \to B_q$ onto the morphism $Ff : FA_{Fp} \to FB_{Fq}$. The functoriality of $\widehat{F}$ follows directly from the functoriality of F. Lastly, a natural transformation $\varphi : F \to G$ is mapped onto the natural transformation $\widehat{\varphi} : \widehat{F} \to \widehat{G}$ which has components $\widehat{\varphi}_{A_p} = G(p) \circ \varphi_A = \varphi_A \circ F(p) : FA_{Fp} \to GA_{Gp}$. The components are morphisms in $\widehat{\mathcal{D}}$ since

$$\widehat{\varphi}_{A_p} \circ F(p) = \varphi_A \circ F(p) \circ F(p) = \varphi_A \circ F(p^2) = \varphi_A \circ F(p) = \widehat{\varphi}_{A_p} \text{ and}$$
$$G(p) \circ \widehat{\varphi}_{A_p} = G(p) \circ G(p) \circ \varphi_A = G(p^2) \circ \varphi_A = G(p) \circ \varphi_A = \widehat{\varphi}_{A_p}.$$

For $\widehat{\varphi}$ to be natural, we need that the square

$$\begin{array}{ccc} FA_{Fp} & \xrightarrow{Ff} & FB_{Fq} \\ \downarrow{\scriptstyle \widehat{\varphi}_{A_p}} & & \downarrow{\scriptstyle \widehat{\varphi}_{B_q}} \\ GA_{Gp} & \xrightarrow{Gf} & GB_{Gq} \end{array}$$

commutes for all morphisms $f : A_p \to B_q$. This follows since the diagram



commutes. We now need to check that $\widehat{(-)}$ is functorial both on functors and natural transformations. Let $\mathcal{C},\mathcal{D},\mathcal{E}$ be categories and $F,G,H:\mathcal{C}\to\mathcal{D}$, $K:\mathcal{D}\to\mathcal{E}$ functors, and $\varphi:F\to G$, $\psi:G\to H$ natural transformations. We have $\widehat{\mathrm{id}_\mathcal{C}}=\mathrm{id}_{\widehat{\mathcal{C}}}$ and also $\widehat{KF}=\widehat{K}\widehat{F}$. For natural transformations we have $(\widehat{\mathrm{id}_F})_{A_p}=F(p)=\mathrm{id}_{FA_{Fp}}=(\mathrm{id}_{\widehat{F}})_{A_p}$ and

$$\widehat{\psi}_{A_p}\widehat{\varphi}_{A_p}=\psi_A G(p)G(p)\varphi_A=\psi_A G(p)\varphi_A=\psi_A\varphi_A F(p)=(\psi\varphi)_A F(p)=\widehat{\psi\varphi}_{A_p}.$$

Thus $\widehat{(-)}$ defines a 2-functor.

To have an adjunction $\widehat{(-)}\vdash\mathcal{U}$, we now need natural transformations $\eta:\mathrm{id}_{\mathbf{Cat}}\to\mathcal{U}\widehat{(-)}$ and $\epsilon:\widehat{(-)}\mathcal{U}\to\mathrm{id}_{\mathbf{Cat}_{ic}}$. These will have components $\eta_\mathcal{C}:\mathcal{C}\to\widehat{\mathcal{C}}$ and $\epsilon_\mathcal{D}:\widehat{\mathcal{D}}\to\mathcal{D}$ for categories $\mathcal{C}$ and idempotent complete categories $\mathcal{D}$.

Since we already defined functors $\iota_\mathcal{C}:\mathcal{C}\to\widehat{\mathcal{C}}$, we define η to have components $\iota_\mathcal{C}$ and denote the natural transformation by ι. We can define ϵ and its components $\epsilon_\mathcal{D}:\widehat{\mathcal{D}}\to\mathcal{D}$ in the following way.

Let A_p be an object in $\widehat{\mathcal{D}}$. Since $\mathcal{D}$ is idempotent complete, we have a splitting (A,B,f,g) of p, i.e., an object B in $\mathcal{D}$ and morphisms $f:A\to B$ and $g:B\to A$ such that $g\circ f=p$ and $f\circ g=\mathrm{id}_B$. To define $\epsilon_\mathcal{D}$, we need to choose a splitting for each idempotent A_p and we choose these such that any identity idempotent A_{id_A} splits via $(A,A,\mathrm{id}_A,\mathrm{id}_A)$. We now map the idempotent A_p onto B. A morphism h between objects A_p and $A'_{p'}$ that have splittings (A,B,f,g) and (A',B',f',g') is mapped onto the morphism $f'\circ h\circ g:B\to B'$.

This assignment is functorial. Let A_p, $A'_{p'}$ and $A''_{p''}$ be objects in $\widehat{\mathcal{D}}$ that have splittings (A,B,f,g), (A',B',f',g') and (A'',B'',f'',g'') and let $h:A_p\to A'_{p'}$ and $h':A'_{p'}\to A''_{p''}$. be morphisms in $\widehat{\mathcal{D}}$. We now have

$\epsilon_\mathcal{D}(\mathrm{id}_{A_p})=\epsilon_\mathcal{D}(p)=f\circ p\circ g=f\circ g\circ f\circ g=\mathrm{id}_B\circ\mathrm{id}_B=\mathrm{id}_B=\mathrm{id}_{\epsilon_\mathcal{D}(A_p)}$ and
$\epsilon_\mathcal{D}(h')\circ\epsilon_\mathcal{D}(h)=f''\circ h'\circ g'\circ f'\circ h\circ g=f''\circ h'\circ p'\circ h\circ g=f''\circ h'\circ h\circ g=\epsilon_\mathcal{D}(h'\circ h).$

ι defines a natural transformation since, for any functor $F:\mathcal{C}\to\mathcal{C}'$ and any object A in $\mathcal{C}$, we have

$$\widehat{F}\iota_\mathcal{C}(A)=\widehat{F}(A_{\mathrm{id}_A})=F(A)_{F(\mathrm{id}_A)}=F(A)_{\mathrm{id}_{F(A)}}=\iota_{\mathcal{C}'}F(A).$$

ϵ is however only natural up to isomorphism since, for an object A_p in $\widehat{\mathcal{D}}$ with a splitting (A,B,f,g) and a functor $F:\mathcal{D}\to\mathcal{D}'$, the splitting we choose for FA_{Fp} does not have to agree with (FA,FB,Ff,Fg). Still, these splittings must be uniquely isomorphic since a splitting of an idempotent is given by a colimit.

Lastly, we need to check that the two triangle identities $(\epsilon\circ\widehat{(-)})\cdot(\widehat{(-)}\circ\iota)=\mathrm{id}_{\widehat{(-)}}$ and $(\mathcal{U}\circ\epsilon)\cdot(\iota\circ\mathcal{U})=\mathrm{id}_{\mathcal{U}}$ hold. By looking at their components, we see that these identities translate to $\epsilon_{\widehat{\mathcal{C}}}\iota_{\widehat{\mathcal{C}}}=\mathrm{id}_{\widehat{\mathcal{C}}}$ and $\epsilon_{\mathcal{D}}\iota_{\mathcal{D}}=\mathrm{id}_{\mathcal{D}}$, which means we just have to check that $\epsilon_{\mathcal{D}}\iota_{\mathcal{D}}=\mathrm{id}_{\mathcal{D}}$ holds for any idempotent complete category $\mathcal{D}$. Since we defined $\epsilon_{\mathcal{D}}$ by choosing that an identity idempotent splits via the identity, this holds automatically. □

Corollary 1.13. *For each category $\mathcal{C}$ and idempotent complete category $\mathcal{D}$, we have an equivalence of categories $\mathrm{Cat}(\widehat{\mathcal{C}},\mathcal{D})\simeq\mathrm{Cat}(\mathcal{C},\mathcal{D})$ induced by precomposing with $\iota_{\mathcal{C}}$.*

Proof. The adjunction $\widehat{(-)}\dashv\mathcal{U}$ induces an equivalence $\mathbf{Cat}_{\mathrm{ic}}(\widehat{\mathcal{C}},\mathrm{D})\simeq\mathbf{Cat}(\mathcal{C},\mathcal{U}(\mathcal{D}))$ by mapping $F:\widehat{\mathcal{C}}\to\mathcal{D}$ onto $\mathcal{U}(F)\iota_{\mathcal{C}}$. Since $\mathcal{U}$ is a forgetful functor we have $\mathcal{U}(\mathcal{D})=\mathcal{D}$ and $\mathcal{U}(F)=F$. The corollary now follows since $\mathbf{Cat}_{\mathrm{ic}}$ is a full subcategory of $\mathbf{Cat}$. □

1.3 1-Categorical Excursion

We have already seen that a splitting of an idempotent is equivalent to the data of a colimit which is preserved by every functor with appropriate domain. Before we can further investigate these types of colimits, we must first reflect on what it means for a functor to preserve a colimit.

Lemma 1.14. *Let $F:\mathcal{J}\to\mathcal{C}$ and $G:\mathcal{C}\to\mathcal{D}$ be functors and let $\lambda^F:F\to\Delta_{\mathrm{colim}F}$ be a colimit cone for F where Δ_- denotes the constant functor at the specified object. The following three notions of* "G preserves colimF" *are equivalent.*

(i) G *maps the colimit cone* λ^F *onto a colimit cone* $G\lambda^F:GF\to\Delta_{G\mathrm{colim}F}$.

(ii) G *maps the natural isomorphism given by precomposing with* λ^F

$$(\lambda^F)^*:\mathcal{C}(\mathrm{colim}F,-)\to\mathrm{Cat}(\mathcal{J},\mathcal{C})(F,\Delta_-)$$

onto a natural isomorphism

$$(G\lambda^F)^*:\mathcal{D}(G\mathrm{colim}F,-)\to\mathrm{Cat}(\mathcal{J},\mathcal{D})(GF,\Delta_-).$$

(iii) *Let $\lambda^{GF}:GF\to\Delta_{\mathrm{colim}GF}$ be a colimit cone for GF. The canonical morphism $\varphi:\mathrm{colim}GF\to G\mathrm{colim}F$, which is given by $\varphi\circ\lambda^{GF}=G\lambda^F$, is an isomorphism.*

Proof. The statements (i) and (ii) are equivalent since $\lambda\colon F\to\Delta_X$ is a colimit cone for a given functor $F\colon\mathcal{J}\to\mathcal{C}$ iff $\lambda^*\colon\mathcal{C}(X,-)\to\mathbf{Cat}(\mathcal{J},\mathcal{C})(F,\Delta_-)$ is a natural isomorphism.

Now, assume (ii) holds. Alongside the canonical morphism $\varphi\colon\mathrm{colim}GF\to G\mathrm{colim}F$, we also get a canonical morphism $\psi\colon G\mathrm{colim}F\to\mathrm{colim}GF$ defined by $\psi\circ G\lambda^F=\lambda^{GF}$. We now have $\varphi\circ\psi\circ G\lambda^F=G\lambda^F$ and $\psi\circ\varphi\circ\lambda^{GF}=\lambda^{GF}$. Since these morphisms must be unique, we have $\varphi\circ\psi=\mathrm{id}_{G\mathrm{colim}F}$ and $\psi\circ\varphi=\mathrm{id}_{\mathrm{colim}GF}$ and φ is an isomorphism.

Finally, assume (iii) holds. We have $(G\lambda^F)^*=(\varphi\circ\lambda^{GF})^*=(\lambda^{GF})^*\circ\varphi^*$. Since both

$$\varphi^*\colon\mathcal{D}(G\mathrm{colim}F,-)\to\mathcal{D}(\mathrm{colim}GF,-)\text{ and}$$
$$(\lambda^{GF})^*\colon\mathcal{D}(\mathrm{colim}GF,-)\to\mathbf{Cat}(\mathcal{J},\mathcal{D})(GF,\Delta_-)$$

are natural isomorphisms, $(G\lambda^F)^*$ is also a natural isomorphism. □

We will come to find the third notion of preserving a colimit especially useful as we can sometimes explicitly construct the morphism φ, which then immediately has to be an isomorphism.

Definition 1.15. (Absolute Colimit) Let $\mathcal{J}$, $\mathcal{C}$ be categories and $F\colon\mathcal{J}\to\mathcal{C}$ a functor. A colimit of F is called an *absolute colimit* if for every category $\mathcal{D}$ and functor $G\colon\mathcal{C}\to\mathcal{D}$, it is preserved by G.

Example 1.16. The splitting of an idempotent defines an abolute colimit.

Definition 1.17. (Absolute Category) A category $\mathcal{J}$ is called an *absolute category* if for every category $\mathcal{C}$ and functor $F\colon\mathcal{J}\to\mathcal{C}$, the colimit of F is absolute if it exists.

Example 1.18. We have already seen that $\clubsuit_1$ defines an absolute category since a colimit of a functor out of $\clubsuit_1$ is given by a splitting of an idempotent. Another, yet trivial, example is the terminal category $\mathbb{1}$ with one object and no non-identity morphisms.

Definition 1.19. (Completeness under Absolute Colimits) A category $\mathcal{C}$ is called *complete under absolute colimits* if for every absolute category $\mathcal{J}$ and every functor $F\colon\mathcal{J}\to\mathcal{C}$ the colimit of F exists.

This definition of completeness might exclude some absolute colimits but we will see that even those absolute colimits that do not stem from absolute categories can be expressed non-trivially via absolute categories.

1.4 Cauchy Completion

We will now define the Cauchy completion of a category, which is the completion under absolute colimits. It too is universal among all categories that have this property.

Definition 1.20. (Cauchy completion) We call an object A in a category $\mathcal{C}$ *tiny* if the functor $\mathcal{C}(A,-):\mathcal{C}\to\mathbf{Set}$ is cocontinuous, i.e., preserves all colimits.

Let $\mathcal{C}$ be a category. The *Cauchy completion* of $\mathcal{C}$ is defined to be the the full subcategory of tiny objects in the category $\mathbf{Psh}(\mathcal{C})$ of presheaves on $\mathcal{C}$, i.e., an object in this category is a functor $S:\mathcal{C}\to\mathbf{Set}^{\mathrm{op}}$ such that $\mathbf{Psh}(\mathcal{C})(S,-):\mathbf{Psh}(\mathcal{C})\to\mathbf{Set}$ is cocontinuous. We will denote this category as $\mathbf{Psh}^{\mathrm{tn}}(\mathcal{C})$.

We first will show that $\mathbf{Psh}^{\mathrm{tn}}(\mathcal{C})$ is a completion of $\mathcal{C}$ in the sense that $\mathcal{C}$ embeds into $\mathbf{Psh}^{\mathrm{tn}}(\mathcal{C})$ and is equivalent to it if $\mathcal{C}$ was already complete under absolute colimits. Before we can do this, we will show the following propositions.

Proposition 1.21. *Representable presheaves are tiny.*

Proof. Let A be an object in a category $\mathcal{C}$, we will now show that $\mathcal{C}(-,A)$ is tiny. Let $F:\mathcal{J}\to\mathbf{Psh}(\mathcal{C})$ be a functor with colimit $\mathrm{colim}F$, i.e., we have a natural isomorphism

$$\lambda^*:\mathbf{Psh}(\mathcal{C})(\mathrm{colim}F,-)\to\mathbf{Cat}(\mathcal{J},\mathbf{Psh}(\mathcal{C}))(F,\Delta_-)$$

given by precomposing with a colimit cone $\lambda:F\to\Delta_{\mathrm{colim}F}$. We will now see that $\mathbf{Psh}(\mathcal{C})(\mathcal{C}(-,A),-)$ preserves this colimit. Applying this functor to λ yields a cone

$$\mathbf{Psh}(\mathcal{C})(\mathcal{C}(-,A),\lambda):\mathbf{Psh}(\mathcal{C})(\mathcal{C}(-,A),F)\to\Delta_{\mathbf{Psh}(\mathcal{C})(\mathcal{C}(-,A),\mathrm{colim}F)}$$

which is defined by component-wise postcomposition with λ. We will now check that

$$\begin{aligned}\mathbf{Psh}(\mathcal{C})(\mathcal{C}(-,A),\lambda)^*:\mathbf{Set}(&\mathbf{Psh}(\mathcal{C})(\mathcal{C}(-,A),\mathrm{colim}F),-)\\ &\to\mathbf{Cat}(\mathcal{J},\mathbf{Set})(\mathbf{Psh}(\mathcal{C})(\mathcal{C}(-,A),F),\Delta_-)\end{aligned}$$

defines a natural isomorphism. We know that it is natural therefore we only have to check that its components are isomorphisms. This translates to the statement that, for each set X and morphism $\varphi:\mathbf{Psh}(\mathcal{C})(\mathcal{C}(-,A),F)\to\Delta_X$, there exists a unique morphism $\psi:\mathbf{Psh}(\mathcal{C})(\mathcal{C}(-,A),\mathrm{colim}F)\to X$ such that the diagram

$$\begin{array}{ccc}\mathbf{Psh}(\mathcal{C})(\mathcal{C}(-,A),F) & \xrightarrow{\varphi} & \Delta_X\\ {\scriptstyle \mathbf{Psh}(\mathcal{C})(\mathcal{C}(-,A),\lambda)}\downarrow & \nearrow_{\psi} & \\ \Delta_{\mathbf{Psh}(\mathcal{C})(\mathcal{C}(-,A),\mathrm{colim}F)} & & \end{array}$$

commutes. Using the Yoneda lemma, we can extend this diagram to the following.

$$\begin{array}{ccccc} & & \overset{\varphi}{\curvearrowright} & & \\ \mathbf{Psh}(\mathcal{C})(\mathcal{C}(-,A),F) & \xrightarrow[\cong]{\mathcal{Y}} & F(-)(A) & & \Delta_X \\ \Big\downarrow{\scriptstyle \mathbf{Psh}(\mathcal{C})(\mathcal{C}(-,A),\lambda)} & & \Big\downarrow{\scriptstyle (\lambda_-)_A} & \nearrow{\scriptstyle \widetilde{\psi}} & \\ \Delta_{\mathbf{Psh}(\mathcal{C})(\mathcal{C}(-,A),\mathrm{colim}F)} & \xrightarrow[\cong]{\mathcal{Y}} & \Delta_{\mathrm{colim}F(A)} & & \end{array}$$

Since colimits in $\mathbf{Psh}(\mathcal{C})$ are computed point-wise, $(\lambda_-)_A$ defines a colimit cone for the functor $F(-)(A):\mathcal{J}\to\mathbf{Set}$ and we get a unique morphism $\widetilde{\psi}:\mathrm{colim}F(A)\to X$ such that $\widetilde{\psi}\circ(\lambda_-)_A=\varphi\circ\mathcal{Y}^{-1}$. Now we just need to confirm that the left square commutes. For this, we check that it commutes evaluated at an arbitrary object j in $\mathcal{J}$.

$$\begin{array}{ccc} \mathbf{Psh}(\mathcal{C})(\mathcal{C}(-,A),Fj) & \xrightarrow{\mathcal{Y}} & F(j)(A) \\ \Big\downarrow{\scriptstyle (\lambda_j)_*} & & \Big\downarrow{\scriptstyle (\lambda_j)_A} \\ \mathbf{Psh}(\mathcal{C})(\mathcal{C}(-,A),\mathrm{colim}F) & \xrightarrow{\mathcal{Y}} & \mathrm{colim}F(A) \end{array}$$

Let $\alpha:\mathcal{C}(-,A)\to Fj$ be a natural transformation. We now have

$$(\mathcal{Y}\circ(\lambda_j)_*)(\alpha)=\mathcal{Y}(\lambda_j\circ\alpha)=(\lambda_j\circ\alpha)_A(\mathrm{id}_A)=(\lambda_j)_A(\alpha_A(\mathrm{id}_A))=((\lambda_j)_A\circ\mathcal{Y})(\alpha)$$

and therefore the square commutes. We now set $\psi=\widetilde{\psi}\circ\mathcal{Y}$. This satisfies

$$\psi\circ\mathbf{Psh}(\mathcal{C})(\mathcal{C}(-,A),\lambda)=\widetilde{\psi}\circ\mathcal{Y}\circ\mathbf{Psh}(\mathcal{C})(\mathcal{C}(-,A),\lambda)=\widetilde{\psi}\circ(\lambda_-)_A\circ\mathcal{Y}=\varphi\circ\mathcal{Y}^{-1}\circ\mathcal{Y}=\varphi$$

and is furthermore unique since $\widetilde{\psi}$ was unique. Thus $\mathcal{C}(-,A)$ is tiny. □

Proposition 1.22. *Every tiny presheaf S on $\mathcal{C}$ is a retract of a representable presheaf.*

Proof. Let S be a tiny presheaf on $\mathcal{C}$. The co-Yoneda Lemma, otherwise also known as the density theorem, states that each presheaf is a colimit of representables. This means there exists a category $\mathcal{J}$, a functor $F:\mathcal{J}\to\mathcal{C}$ and a colimit cone $\lambda:\text{よ}_{\mathcal{C}}F\to\Delta_S$.

Since S is tiny, applying $\mathbf{Psh}(\mathcal{C})(S,-)$ yields another colimit cone

$$\mathbf{Psh}(\mathcal{C})(S,\lambda):\mathbf{Psh}(\mathcal{C})(S,\text{よ}_{\mathcal{C}}F)\to\Delta_{\mathbf{Psh}(\mathcal{C})(S,S)}.$$

Since the functor $\mathbf{Psh}(\mathcal{C})(S,\text{よ}_{\mathcal{C}}F):\mathcal{J}\to\mathbf{Set}$ takes values in $\mathbf{Set}$, we can explicitly construct a second colimit cone. We define a relation on the set

$$\coprod_{j\in\mathrm{Ob}\mathcal{J}}\mathbf{Psh}(\mathcal{C})(S,\mathcal{C}(-,Fj))$$

in the following way. For each $\alpha: S \to \mathcal{C}(-, Fj_1)$ and $\beta: S \to \mathcal{C}(-, Fj_2)$, we set $\alpha \sim \beta$ if there exists a morphism $f: j_1 \to j_2$ in $\mathcal{J}$ such that $F(f)_* \circ \alpha = \beta$. We now take the equivalence relation generated by this relation and define colim$\mathbf{Psh}(\mathcal{C})(S, \text{よ}_\mathcal{C} F)$ to be the set of equivalence classes under this relation. The cone

$$\widetilde{\lambda}: \mathbf{Psh}(\mathcal{C})(S, \text{よ}_\mathcal{C} F) \to \Delta_{\mathrm{colim}\mathbf{Psh}(\mathcal{C})(S, \text{よ}_\mathcal{C} F)}$$

is now defined via $\widetilde{\lambda}_j(\alpha) = [\alpha]$, i.e., for each j in $\mathcal{J}$, $\widetilde{\lambda}_j$ maps $\alpha: S \to \mathcal{C}(-, F(j))$ onto its equivalence class $[\alpha]$. We now also have a unique map

$$\varphi: \mathrm{colim}\mathbf{Psh}(\mathcal{C})(S, \text{よ}_\mathcal{C} F) \to \mathbf{Psh}(\mathcal{C})(S, S)$$

such that $\varphi \circ \widetilde{\lambda} = \mathbf{Psh}(\mathcal{C})(S, \lambda)$, which is defined via $\varphi([\alpha]) = \lambda_j \circ \alpha$.

We know that this map has to be an isomorphism, which implies that there exists a j in $\mathcal{J}$ and an $\alpha: S \to \mathcal{C}(-, Fj)$ such that $\lambda_j \circ \alpha = \mathrm{id}_S$. Therefore $\alpha \circ \lambda_j$ is an idempotent on $\mathcal{C}(-, Fj)$, which splits via S. □

Proposition 1.23. *For every category $\mathcal{C}$, the Yoneda embedding $\text{よ}_\mathcal{C}: \mathcal{C} \to \mathrm{Psh}(\mathcal{C})$ takes values in $\mathrm{Psh}^{\mathrm{tn}}(\mathcal{C})$ and thus defines a fully faithful functor $\text{よ}_\mathcal{C}: \mathcal{C} \to \mathrm{Psh}^{\mathrm{tn}}(\mathcal{C})$. If $\mathcal{C}$ is furthermore complete under absolute colimits, this functor is an equivalence.*

Proof. Let A be an object in $\mathcal{C}$. By proposition 1.21, we have that $\text{よ}_\mathcal{C}(A) = \mathcal{C}(-, A)$ is tiny. Thus the Yoneda embedding takes values in tiny presheaves and we have a fully faithful functor $\text{よ}_\mathcal{C}: \mathcal{C} \to \mathbf{Psh}^{\mathrm{tn}}(\mathcal{C})$.

Now assume that $\mathcal{C}$ is complete under absolute colimits. Let S be a tiny presheaf on $\mathcal{C}$. By proposition 1.22, there is an object A in $\mathcal{C}$ and idempotent $p: \mathcal{C}(-, A) \to \mathcal{C}(-, A)$ such that p splits via S. Since the Yoneda embedding is fully faithful, there is a morphism $\widetilde{p}: A \to A$ such that $\widetilde{p}_* = p$.

Since $\mathcal{C}$ is assumed to be complete under absolute colimits and a splitting of an idempotent is a colimit of a functor out of an absolute category, there is an object B in $\mathcal{C}$ such that $\widetilde{p}$ splits via B. But now the idempotent $\widetilde{p}_* = p$ also splits via $\mathcal{C}(-, B)$ and since colimits are unique up to isomorphism we have $S \cong \mathcal{C}(-, B)$. Therefore the functor $\text{よ}_\mathcal{C}: \mathcal{C} \to \mathbf{Psh}^{\mathrm{tn}}(\mathcal{C})$ is essentially surjective and thus an equivalence of categories. □

We can now also see in what way a general absolute colimit is related to absolute categories.

Lemma 1.24. *Let $\mathcal{J}$ and $\mathcal{C}$ be categories, $F: \mathcal{J} \to \mathcal{C}$ a functor and let $\lambda: F \to \Delta_A$ define an absolute colimit for some A in $\mathcal{C}$. A is a retract of an object in the image of F.*

Proof. If we apply the functor $\mathcal{C}(A,-):\mathcal{C}\to\mathbf{Set}$ to the given colimit, we get a colimiting cone $\mathcal{C}(A,\lambda):\mathcal{C}(A,F)\to\Delta_{\mathcal{C}(A,A)}$. Since $\mathcal{C}(A,F)$ is now a functor into $\mathbf{Set}$, we have, analogously to the proof of proposition 1.22, that A is a retract of Fj for some j in $\mathcal{J}$. This means we have a functor $F':\clubsuit_1\to\mathcal{C}$ and a colimit cone $\lambda':F'\to\Delta_A$. □

We furthermore conjecture that the idempotent itself also lies in the image of F.

Next we will check that the Cauchy completion of a category is indeed complete under absolute colimits, which we will do using the following lemma.

Lemma 1.25. *A retract of a tiny object is tiny.*

Proof. Let B be a retract of a tiny object A in a category $\mathcal{C}$, i.e., we have morphisms $f:A\to B$ and $g:B\to A$ such that $f\circ g=\mathrm{id}_B$. Let $F:\mathcal{J}\to\mathcal{C}$ be a functor with colimit cone $\lambda:F\to\Delta_C$. We will need to show that $\mathcal{C}(B,\lambda):\mathcal{C}(B,F)\to\Delta_{\mathcal{C}(B,C)}$ also defines a colimit cone, which is equivalent to

$$\mathcal{C}(B,\lambda)^*:\mathbf{Set}(\mathcal{C}(B,C),-)\to\mathbf{Cat}(\mathcal{J},\mathbf{Set})(\mathcal{C}(B,F),\Delta_-)$$

being a natural isomorphism. Since it is automatically natural, we will just need to show that it is an isomorphism in every component. Let $\varphi:\mathcal{C}(B,F)\to\Delta_X$ be a natural transformation, we need to show that there is a unique $\psi:\mathcal{C}(B,C)\to X$ such that the diagram

$$\begin{array}{ccc} \mathcal{C}(B,F) & \xrightarrow{\varphi} & \Delta_X \\ \downarrow{\scriptstyle\lambda_*} & \nearrow{\scriptstyle\psi} & \\ \Delta_{\mathcal{C}(B,C)} & & \end{array}$$

commutes. We can expand this diagram in the following way.

$$\begin{array}{ccccc} \mathcal{C}(A,F) & \underset{f^*}{\overset{g^*}{\rightleftarrows}} & \mathcal{C}(B,F) & \xrightarrow{\varphi} & X \\ \downarrow{\scriptstyle\lambda_*} & & \downarrow{\scriptstyle\lambda_*} & \nearrow{\scriptstyle\psi} & \\ \Delta_{\mathcal{C}(A,C)} & \underset{f^*}{\overset{g^*}{\rightleftarrows}} & \Delta_{\mathcal{C}(B,C)} & & \end{array}$$

Since A is tiny, we know that $\mathcal{C}(A,\lambda):\mathcal{C}(A,F)\to\Delta_{\mathcal{C}(A,C)}$ is a colimiting cone. We can now define $\widetilde{\varphi}=\varphi\circ g^*$ and thus have a unique $\widetilde{\psi}:\mathcal{C}(A,C)\to X$ such that $\widetilde{\psi}\circ\lambda_*=\widetilde{\varphi}$. We now set $\psi=\widetilde{\psi}\circ f^*$ and have

$$\psi\circ\lambda_*=\widetilde{\psi}\circ f^*\circ\lambda_*=\widetilde{\psi}\circ\lambda_*\circ f^*=\widetilde{\varphi}\circ f^*=\varphi\circ g^*\circ f^*=\varphi\circ(f\circ g)^*=\varphi.$$

ψ is furthermore uniquely determined, since $\widetilde{\psi}$ was uniquely determined. Thus $\mathcal{C}(B,\lambda)$ defines a colimit cone and B is a tiny object. □

Proposition 1.26. *For any category* $\mathcal{C}$*, its Cauchy completion* $\mathrm{Psh}^{\mathrm{tn}}(\mathcal{C})$ *is complete under absolute colimits.*

Proof. Let $\mathcal{J}$ be an absolute category and $F\colon\mathcal{J}\to\mathbf{Psh}^{\mathrm{tn}}(\mathcal{C})$ a functor. A priori, the colimit of F does not need to exist in $\mathbf{Psh}^{\mathrm{tn}}(\mathcal{C})$ but since $\mathbf{Psh}(\mathcal{C})$ is cocomplete, it will exist in $\mathbf{Psh}(\mathcal{C})$. Let $\lambda\colon F\to\Delta_S$ be a colimit cone for some S in $\mathbf{Psh}(\mathcal{C})$. By lemma 1.24, S is a retract of a tiny presheaf and thus, by lemma 1.25, S is a tiny presheaf, proving that $\mathbf{Psh}^{\mathrm{tn}}(\mathcal{C})$ is complete under absolute colimits. □

We would now like to show that Cauchy completion is also universal among all completions under absolute colimits. We will prove this by showing that the Cauchy completion and the Karoubi completion are equivalent constructions, i.e., they give equivalent categories.

Theorem 1.27. *Let* $\mathcal{C}$ *be a category. The functor given by the composition*

$$\widehat{\mathcal{C}} \xrightarrow{\ \text{よ}_{\widehat{\mathcal{C}}}\ } \mathrm{Psh}(\widehat{\mathcal{C}}) \xrightarrow{\ \iota_{\mathcal{C}}^{*}\ } \mathrm{Psh}(\mathcal{C})$$

defines an equivalence of categories $\widehat{\mathcal{C}}\simeq\mathrm{Psh}^{\mathrm{tn}}(\mathcal{C})$*.*

Proof. First, we want to show that the functor takes values in tiny objects, i.e., for every object A_p in $\widehat{\mathcal{C}}$, the presheaf $\widehat{\mathcal{C}}(\iota_{\mathcal{C}},A_p)\colon\mathcal{C}\to\mathbf{Set}^{\mathrm{op}}$ is tiny. By remark 1.11, A_p is a splitting of the idempotent p on A_{id_A}. By absoluteness of splittings, we have that $\widehat{\mathcal{C}}(\iota_{\mathcal{C}},A_p)$ is a retract of $\widehat{\mathcal{C}}(\iota_{\mathcal{C}},A_{\mathrm{id}_A})=\widehat{\mathcal{C}}(\iota_{\mathcal{C}},\iota_{\mathcal{C}}A)$. Since $\iota_{\mathcal{C}}$ is fully faithful, we have $\widehat{\mathcal{C}}(\iota_{\mathcal{C}},\iota_{\mathcal{C}}A)\cong\mathcal{C}(-,A)$. This means that $\widehat{\mathcal{C}}(\iota_{\mathcal{C}},A_p)$ is a retract of a representable presheaf and by lemma 1.25, it therefore has to be tiny.

Next, we will show that the functor is fully faithful. Since the Yoneda embedding is fully faithful and by corollary 1.13 with $\mathcal{D}=\mathbf{Set}^{\mathrm{op}}$, precomposition with $\iota_{\mathcal{C}}$ is fully faithful, their composition must also be fully faithful.

Lastly, we need that the functor is essentially surjective. Let S be a tiny presheaf on $\mathcal{C}$. By proposition 1.22, there exists an object A in $\mathcal{C}$ and an idempotent p on A such that S is a splitting of the idempotent p_* on $\mathcal{C}(-,A)$. By remark 1.11, A_p is a splitting of the idempotent p on $A_{\mathrm{id}_A}=\iota_{\mathcal{C}}A$. By absoluteness of splittings, $\widehat{\mathcal{C}}(\iota_{\mathcal{C}},A_p)$ is a splitting of the idempotent p_* on $\widehat{\mathcal{C}}(\iota_{\mathcal{C}},\iota_{\mathcal{C}}A)\cong\mathcal{C}(-,A)$. Since splittings of idempotents are unique up to isomorphisms, it follows that $S\cong\widehat{\mathcal{C}}(\iota_{\mathcal{C}},A_p)$. □

It now follows that the notions of idempotent completeness and completeness under absolute colimits are equivalent.

Corollary 1.28. *A category is idempotent complete iff it is complete under absolute colimits.*

Proof. Let $\mathcal{C}$ be an idempotent complete category. By proposition 1.9 and theorem 1.27, we now have $\mathcal{C} \simeq \widehat{\mathcal{C}} \simeq \mathbf{Psh}^{\mathrm{tn}}(\mathcal{C})$. Since $\mathbf{Psh}^{\mathrm{tn}}(\mathcal{C})$ is complete under absolute colimits, $\mathcal{C}$ also must be. The opposite direction follows analogously with proposition 1.23. ☐

Finally we can show that the Cauchy completion is universal among all completions under absolute colimits in the sense that every functor $F : \mathcal{C} \to \mathcal{D}$ from an arbitrary category into a category complete under absolute colimits, i.e., an idempotent complete category, factors through the Yoneda embedding $よ_{\mathcal{C}} : \mathcal{C} \to \mathbf{Psh}^{\mathrm{tn}}(\mathcal{C})$.

Corollary 1.29. *For every category $\mathcal{C}$ and idempotent complete category $\mathcal{D}$ there is an equivalence*

$$\mathrm{Cat}(\mathrm{Psh}^{\mathrm{tn}}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Cat}(\mathcal{C}, \mathcal{D})$$

given by precomposing with $よ_{\mathcal{C}}$.

Proof. Corollary 1.13 and theorem 1.27 give us equivalences

$$\mathbf{Cat}(\mathbf{Psh}^{\mathrm{tn}}(\mathcal{C}), \mathcal{D}) \xrightarrow{(\iota_{\mathcal{C}}^* よ_{\widehat{\mathcal{C}}})^*} \mathbf{Cat}(\widehat{\mathcal{C}}, \mathcal{D}) \xrightarrow{\iota_{\mathcal{C}}^*} \mathbf{Cat}(\mathcal{C}, \mathcal{D}).$$

Composing them we get $\iota_{\mathcal{C}}^*(\iota_{\mathcal{C}}^* よ_{\widehat{\mathcal{C}}})^* = (\iota_{\mathcal{C}}^* よ_{\widehat{\mathcal{C}}} \iota_{\mathcal{C}})^*$. For an object A in $\mathcal{C}$, we get $\iota_{\mathcal{C}}^* よ_{\widehat{\mathcal{C}}} \iota_{\mathcal{C}} A = \widehat{\mathcal{C}}(\iota_{\mathcal{C}}, \iota_{\mathcal{C}} A) \cong \mathcal{C}(-, A) = よ_{\mathcal{C}} A$ and therefore $\iota_{\mathcal{C}}^*(\iota_{\mathcal{C}}^* よ_{\widehat{\mathcal{C}}})^* \cong よ_{\mathcal{C}}^*$. ☐

2 Bicategorical Prerequisites

We will now lay down the prerequisites in bicategory theory required for the rest of this thesis and retroactively supplying the definitions needed to make theorem 1.12 rigorous. We try to stick to the terminology used in [JY21].

2.1 Bicategorical Fundamentals

This first section roughly follows chapters 2 and 4 of [JY21] which gives an extensive introduction to the theory of bicategories. To begin, we will give the definition of a bicategory.

Definition 2.1. (Bicategory) A *bicategory* $\mathbb{B}$ consists of the following data.

- A collection of *objects* $\mathrm{Ob}\mathbb{B}$. We will often times abbreviate $A \in \mathrm{Ob}\mathbb{B}$ as $A \in \mathbb{B}$ for an object A in $\mathbb{B}$.
- For each pair of objects $A, B \in \mathbb{B}$, a category $\mathbb{B}(A,B)$ called a *Hom-category*. Objects in $\mathbb{B}(A,B)$ are called *1-morphisms* and morphisms between 1-morphisms are called *2-morphisms*. Composition of 2-morphisms is called *vertical composition*.
- For each object $A \in \mathbb{B}$, a 1-morphism $\mathrm{id}_A \in \mathbb{B}(A,A)$, called the *identity* on A.
- For each triple of objects $A, B, C \in \mathbb{B}$, a functor

  $$c_{ABC} : \mathbb{B}(B,C) \times \mathbb{B}(A,B) \to \mathbb{B}(A,C),$$

  called the *horizontal composition*. For 1-morphisms $f, f' \in \mathbb{B}(A,B)$, $g, g' \in \mathbb{B}(B,C)$ and 2-morphisms $\alpha : f \to f'$ and $\beta : g \to g'$ we write

  $$c_{ABC}(g,f) = g \circ f \in \mathbb{B}(A,C),$$
  $$c_{ABC}(\beta,\alpha) = \beta \circ \alpha : g \circ f \to g' \circ f'.$$

  When it is clear from context, we will sometimes notate horizontal composition by concatenation and leave out the symbol for composition.
- For each collection of objects $A, B, C, D \in \mathbb{B}$, a natural isomorphism

  $$a_{ABCD} : c_{ABD}(c_{ABC} \times \mathrm{id}_{\mathbb{B}(A,B)}) \to c_{ACD}(\mathrm{id}_{\mathbb{B}(C,D)} \times c_{ABC}),$$

  called the *associator*, between functors

  $$\mathbb{B}(C,D) \times \mathbb{B}(B,C) \times \mathbb{B}(A,B) \to \mathbb{B}(A,D).$$

- For each pair of object $A,B \in \mathbb{B}$ a natural isomorphism

$$l_{AB} : c_{ABB}(id_B, -) \to id_{\mathbb{B}(A,B)},$$

  called the *left unitor*.

- For each pair of object $A,B \in \mathbb{B}$ a natural isomorphism

$$r_{AB} : c_{AAB}(-, id_A) \to id_{\mathbb{B}(A,B)},$$

  called the *right unitor*.

The subscripts in c will often be omitted. The subscripts in a, l and r will often be used to denote their component morphisms. This data needs to satisfy the two following axioms for all 1-morphisms $f \in \mathbb{B}(A,B)$, $g \in \mathbb{B}(B,C)$, $h \in \mathbb{B}(C,D)$ and $k \in \mathbb{B}(D,E)$.

Triangle Axiom: The diagram

$$\begin{array}{ccc} (g \circ id_B) \circ f & \xrightarrow{\quad a \quad} & g \circ (id_B \circ f) \\ & {\scriptstyle r_g \circ id_f} \searrow \qquad \swarrow {\scriptstyle id_g \circ l_f} & \\ & g \circ f & \end{array}$$

in $\mathbb{B}(A,C)$ commutes.

Pentagon Axiom: The diagram

$$\begin{array}{ccccc} & & (k \circ h) \circ (g \circ f) & & \\ & \nearrow^{a_{k \circ h, g, f}} & & \searrow^{a_{k, h, g \circ f}} & \\ ((k \circ h) \circ g) \circ f & & & & k \circ (h \circ (g \circ f)) \\ \downarrow {\scriptstyle a_{k,h,g} \circ id_f} & & & & \uparrow {\scriptstyle id_k \circ a_{h,g,f}} \\ (k \circ (h \circ g)) \circ f & & \xrightarrow{\qquad a_{k, h \circ g, f} \qquad} & & k \circ ((h \circ g) \circ f) \end{array}$$

in $\mathbb{B}(A,E)$ commutes.

This finishes the definition of a bicategory.

A special class of bicategories which are simpler to handle are the following.

Definition 2.2. (2-Category) A *2-category* is a bicategory $\mathbb{B}$ in which the associator and the left and right unitor are all identity natural transformations.

Since 2-categories lack some of the complexity of general bicategories, they are easier to work with and one might expect this to limit their usefulness to only few cases. But in fact we will later see that every bicategory is equivalent to a 2-category.

Within a given bicategory the notion of two objects being isomorphic is often times too strong. Thus we define the following.

Definition 2.3. (Equivalence) Let $\mathbb{B}$ be a bicategory, we call two objects A,B in $\mathbb{B}$ *equivalent* if there exist 1-morphisms $f:A\to B$ and $g:B\to A$ such that $g\circ f\cong \mathrm{id}_A$ and $f\circ g\cong \mathrm{id}_B$. We will denote this as $A\simeq B$ and call such 1-morphisms *equivalences.*

We often times will construct a bicategory from a given one by removing some of its objects. This can be made rigorous by the following.

Definition 2.4. (Subbicategory) Let $\mathbb{B}$ and $\mathbb{B}'$ be bicategories. $\mathbb{B}'$ is a *subbicategory* of $\mathbb{B}$ if the following statements hold.

- The collection $\mathrm{Ob}\mathbb{B}'$ is contained within $\mathrm{Ob}\mathbb{B}$.
- For objects $A,B\in\mathbb{B}'$, $\mathbb{B}'(A,B)$ is a subcategory of $\mathbb{B}(A,B)$.
- The identity on A in $\mathbb{B}'$ is equal to the identity of A in $\mathbb{B}$.
- For objects A,B,C in $\mathbb{B}$, the horizontal composition c'_{ABC} in $\mathbb{B}'$ makes the diagram

$$\begin{array}{ccc} \mathbb{B}'(B,C)\times\mathbb{B}'(A,B) & \xrightarrow{c'_{ABC}} & \mathbb{B}'(A,C) \\ \downarrow & & \downarrow \\ \mathbb{B}(B,C)\times\mathbb{B}(A,B) & \xrightarrow{c_{ABC}} & \mathbb{B}(A,C) \end{array}$$

  commute, where the unlabeled arrows are subcategory inclusions.
- Every component of the associator in $\mathbb{B}'$ is equal to the corresponding component of the associator in $\mathbb{B}$ and analogously for the left and right unitors.

Furthermore $\mathbb{B}'$ is a *full subbicategory* of $\mathbb{B}$ if for objects $A,B\in\mathbb{B}'$, $\mathbb{B}'(A,B)$ equals $\mathbb{B}(A,B)$.

We are now going to list some examples of bicategories and ways to construct bicategories.

Example 2.5. (**Cat**) The archetypal bicategory is the 2-category **Cat** of categories, functors and natural transformations.

- Ob**Cat** is given by the collection of all categories.
- For each pair of categories $\mathcal{C},\mathcal{D}\in\mathbf{Cat}$, the collection of functors $\mathbf{Cat}(\mathcal{C},\mathcal{D})$ from $\mathcal{C}$ into $\mathcal{D}$ form a category with natural transformations as morphisms.
- For each category $\mathcal{C}\in\mathbf{Cat}$, there exists an identity functor $\mathrm{id}_{\mathcal{C}}\in\mathbf{Cat}(\mathcal{C},\mathcal{C})$.
- For each triple of categories $\mathcal{C},\mathcal{D},\mathcal{E}\in\mathbf{Cat}$, composition of functors forms a functor

$$c_{\mathcal{C}\mathcal{D}\mathcal{E}}:\mathbf{Cat}(\mathcal{D},\mathcal{E})\times\mathbf{Cat}(\mathcal{C},\mathcal{D})\to\mathbf{Cat}(\mathcal{C},\mathcal{E}).$$

- Since composition of functors is strictly associative and strictly unital, the associator and the left and right unitors are given by identity natural transformations. The triangle axiom and pentagon axiom therefore also immediately hold.

Remark 2.6. Equivalence in the bicategory **Cat** turns out to be the usual notion of equivalence of categories.

Example 2.7. (Opposite Bicategory) Let $\mathbb{B}$ be a bicategory with horizontal composition c, associator a, and left and right unitors l and r. We can define the bicategory $\mathbb{B}^{op}$ in the following way.

- $\mathbb{B}^{op}$ has the same objects as $\mathbb{B}$.
- Let A,B be objects in $\mathbb{B}^{op}$. The hom category $\mathbb{B}^{op}(A,B)$ is given by $\mathbb{B}(B,A)$.
- Let A,B,C be a triple of objects in $\mathbb{B}^{op}$. The functor

$$c^{op}_{ABC}:\mathbb{B}^{op}(B,C)\times\mathbb{B}^{op}(A,B)\to\mathbb{B}^{op}(A,C)$$

  is defined by

$$c^{op}_{ABC}(g,f)=c_{CBA}(f,g)=f\circ g,$$
$$c^{op}_{ABC}(\beta,\alpha)=c_{CBA}(\alpha,\beta)=\alpha\circ\beta:f\circ g\to f'\circ g'$$

  for 1-morphisms $f,f'\in\mathbb{B}^{op}(A,B)=\mathbb{B}(B,A)$, $g,g'\in\mathbb{B}^{op}(B,C)=\mathbb{B}(C,B)$ and 2-morphisms $\alpha:f\to f'$ and $\beta:g\to g'$. Functorality of c^{op} follows from that of c.

- a^{op} is given by a^{-1}, $l^{op}=r$ and $r^{op}=l$. The triangle and pentagon axiom for $\mathbb{B}^{op}$ immediately follow from those in $\mathbb{B}$.

Example 2.8. (Monoidal Categories) Let $\mathcal{C}$, be a monoidal category with monoidal product $\otimes$ and monoidal unit $\mathbb{1}$. $\mathcal{C}$ can be thought of as a bicategory $\mathbb{C}$ in the following way.

- $\mathbb{C}$ has one object which we will call X.
- The one and only Hom-category is given by $\mathbb{C}(X,X)=\mathcal{C}$.
- The identity 1-morphism on X is given by $\mathbb{1}$.
- Horizontal composition is given by the monoidal product

$$\otimes:\mathcal{C}\times\mathcal{C}\to\mathcal{C}.$$

- The associator and left and right unitors are given by the associator and the left and right unitors of the monoidal category. The triangle and pentagon axiom for monoidal categories then immediately translate to the triangle and pentagon axiom of bicategories.

Example 2.9. (**Bimod**) We can define the category **Bimod** in the following way.

- The objects of **Bimod** are given by associative, unital rings.
- For each pair of rings $R,S\in\mathbf{Bimod}$, the Hom-category $\mathbf{Bimod}(R,S)$ is given by the category of (R,S)-bimodules and bimodule morphisms.
- The identity 1-morphism on a ring R is given by regarding R as an (R,R)-bimodule.
- For a triple of rings R,S,T horizontal composition is given by the tensor product of bimodules

$$\begin{aligned}\otimes_S:\mathbf{Bimod}(S,T)\times\mathbf{Bimod}(R,S)&\to\mathbf{Bimod}(S,T)\\(N,M)&\mapsto M\otimes_S N.\end{aligned}$$

  We note that the tensor product of bimodules exists only up to unique isomorphism but the definition of horizontal composition forces us to choose a specific realisation of the tensor product. This definition involves a choice for every pair of modules. Any choice we make is equivalent to any other, yet a choice nonetheless.

- For rings Q,R,S,T, a (Q,R)-bimodule L, an (R,S)-bimodule M and an (S,T)-bimodule N the components of the associator are given by the canonical isomorphism

$$a_{N,M,L}:L\otimes_R(M\otimes_S N)\to(L\otimes_R M)\otimes_S N,$$

the components of the left unitor by the canonical isomorphism

$$l_M : M \otimes_S S \to M$$

and the components of the right unitor by the canonical isomorphism

$$r_M : R \otimes_R M \to M.$$

- Since tensor products are unique up to unique isomorphism, the triangle axiom and the pentagon axiom have to hold, since both ways of going around the diagrams must be the same unique isomorphism.

Lastly, similarly to how a category can have a property, a bicategory can have that property for each of its Hom-categories.

Definition 2.10. (Local Properties) Suppose P is a property of categories. A bicategory $\mathbb{B}$ is *locally* P if every Hom-category of $\mathbb{B}$ has property P. In particular, $\mathbb{B}$ is

- *locally discrete* if each Hom-category is discrete,
- *locally idempotent complete* if each Hom-category is idempotent complete.

Example 2.11. (Categories) An ordinary category $\mathcal{C}$ gives rise to a locally discrete bicategory and thus a 2-category by regarding every Hom-set as a discrete category.

Having defined bicategories, we are now interest to look at morphisms between bicategories. These can take many shapes but in this thesis we are interested in the following ones.

Definition 2.12. (Pseudofunctor) Let $\mathbb{B}$ be a bicategory with associator a and left and right unitors l and r and let $\mathbb{C}$ be a bicategoty with associator a' and left and right unitors l' and r'. A *pseudofunctor* $F : \mathbb{B} \to \mathbb{C}$ consists of the following data.

- An assignment $F : \mathrm{Ob}\mathbb{B} \to \mathrm{Ob}\mathbb{C}$ on objects.
- For each pair of objects $A, B \in \mathbb{B}$, a functor

  $$F_{AB} : \mathbb{B}(A,B) \to \mathbb{C}(FA, FB).$$

  The subscripts in F on Hom-categories will often be omitted.
- For each object $A \in \mathbb{B}$, an isomorphism

  $$F^0_A : \mathrm{id}_{FA} \to F\mathrm{id}_A.$$

- For each triple of objects $A,B,C \in \mathbb{B}$ a natural isomorphism

$$F^2_{ABC} : c'_{FA,FB,FC}(F_{BC} \times F_{AB}) \to F_{AC} c_{ABC}.$$

The subscripts in F^2 will be used to denote its component morphisms

$$F^2_{g,f} : Fg \circ Ff \to F(g \circ f).$$

The above data is required to make the following three diagrams commute for all 1-morphisms $f \in \mathbb{B}(A,B)$, $g \in \mathbb{B}(B,C)$ and $h \in \mathbb{B}(C,D)$.

Compatibility between associators:

$$\begin{array}{ccc}
(Fh \circ Fg) \circ Ff & \xrightarrow{a'} & Fh \circ (Fg \circ Ff) \\
\downarrow{\scriptstyle F^2_{h,g} \circ id_{Ff}} & & \downarrow{\scriptstyle id_{Fh} \circ F^2_{g,f}} \\
F(h \circ g) \circ Ff & & Fh \circ F(g \circ f) \\
\downarrow{\scriptstyle F^2_{h \circ g, f}} & & \downarrow{\scriptstyle F^2_{h, g \circ f}} \\
F((h \circ g) \circ f) & \xrightarrow{Fa} & F(h \circ (g \circ f))
\end{array}$$

Compatibility between unitors:

$$\begin{array}{ccc}
id_{FB} \circ Ff & \xrightarrow{l'} & Ff \\
{\scriptstyle F^0_B \circ id_{Ff}}\downarrow & & \uparrow{\scriptstyle Fl} \\
Fid_B \circ Ff & \xrightarrow{F^2_{id_B, f}} & F(id_B \circ f)
\end{array}
\qquad
\begin{array}{ccc}
Ff \circ id_{FA} & \xrightarrow{r'} & Ff \\
{\scriptstyle id_{Ff} \circ F^0_A}\downarrow & & \uparrow{\scriptstyle Fr} \\
Ff \circ Fid_A & \xrightarrow{F^2_{f, id_A}} & F(f \circ id_A)
\end{array}$$

A pseudofunctor between 2-categories where F^0 and F^2 are given by identities, is called a *2-functor*.

Example 2.13. (Identity Pesudofunctor) Let $\mathbb{B}$ be a bicategory. The identity on $\mathbb{B}$ defines a pesudofunctor $id_{\mathbb{B}} : \mathbb{B} \to \mathbb{B}$ in the following way.

- The assignment on objects assigns each object in $\mathbb{B}$ to itself.
- The functors on the Hom-categories are the appropriate identity functors.
- $id^0_{\mathbb{B}}$ and $id^2_{\mathbb{B}}$ are given by identities.
- The compatibility diagrams commute trivially.

Example 2.14. (Constant Pseudofunctor) Let $\mathbb{B}$ and $\mathbb{C}$ be bicategories, and let X be an object in $\mathbb{C}$. The pseudofunctor $\Delta_X : \mathbb{B} \to \mathbb{C}$ is defined in the following way.

- Δ_X assigns X to every object in $\mathbb{B}$.

- For each pair of objects A,B in $\mathbb{B}$ the functor $\Delta_X : \mathbb{B}(A,B) \to \mathbb{C}(X,X)$ is defined to be the constant functor at id_X.
- Δ_X^0 is given by the identity on id_X.
- Δ_X^2 is given by $l_{id_X} = r_{id_X}$. The fact that this equality holds follows from the Coherence Theorem for bicategories.
- It also follows, that the compatibility diagrams commute.

We call this pseudofunctor the constant pseudofunctor at X.

Analogously to how we are interested in representable functors in ordinary category theory, we are also interested in representable pseudofunctors for which we first need to show the following.

Remark 2.15. Let $\mathbb{B}$ be a bicategory and $f : A \to B$ a 1-morphism in $\mathbb{B}$ and let X be another object in $\mathbb{B}$. Then precomposition with f defines a functor

$$f^* : \mathbb{B}(B,X) \to \mathbb{B}(A,X).$$

A 1-morphism $g : B \to X$ gets mapped to the morphism $g \circ f : A \to X$ and a 2-morphism $\alpha : g \to g'$ gets mapped onto the 2-morphism $\alpha \circ id_f$. Functoriality follows from the functoriality of horizontal composition in $\mathbb{B}$. Analogously, postcomposition with f defines a functor

$$f_* : \mathbb{B}(X,A) \to \mathbb{B}(X,B).$$

Example 2.16. (Representable Pseudofunctor) Let $\mathbb{B}$ be a bicategory and X an object in $\mathbb{B}$. The pseudofunctor $\mathbb{B}(-,X) : \mathbb{B} \to \mathbf{Cat}^{op}$ is defined in the following way.

- $\mathbb{B}(-,X)$ assigns to an object A in $\mathbb{B}$ the category $\mathbb{B}(A,X)$.
- The functor $\mathbb{B}(-,X)_{AB} : \mathbb{B}(A,B) \to \mathbf{Cat}(\mathbb{B}(B,X),\mathbb{B}(A,X))$ is defined by mapping a 1-morphism $f : A \to B$ to the functor f^*. A two morphism $\alpha : f \to f'$ is mapped onto the natural transformation $\mathbb{B}(\alpha,X) : f^* \to (f')^*$ with components

  $$\mathbb{B}(\alpha,X)_g = id_g \circ \alpha : g \circ f \to g \circ f'.$$

  the naturality of $\mathbb{B}(\alpha,X)$ and functorality of $\mathbb{B}(-,X)_{AB}$ both follow from the functorality of horizontal composition in $\mathbb{B}$.
- The isomorphism $\mathbb{B}(-,X)_A^0 : id_{\mathbb{B}(A,X)} \to \mathbb{B}(id_A,X) = id_A^*$ is given by r^{-1}.

- The isomorphism $\mathbb{B}(-,X)^2_{g,f}:f^*g^* \to (g\circ f)^*$ has components

$$(h\circ g)\circ f \to h\circ(g\circ f)$$

  given by the associator a in $\mathbb{B}$. Naturality follows from the naturality of a.

- It follows from the pentagon and triangle axioms in $\mathbb{B}$, that the compatibility diagrams commute.

Analogously, we get a pseudofunctor $\mathbb{B}(X,-):\mathbb{B}\to\mathbf{Cat}$.

Example 2.17. (Composition of Pseudofunctors) Let $\mathbb{B},\mathbb{C},\mathbb{D}$ be bicategories and let $F:\mathbb{B}\to\mathbb{C}$ and $G:\mathbb{C}\to\mathbb{D}$ be pseudofunctors. The composite $GF:\mathbb{B}\to\mathbb{C}$ defines a pseudofunctor in the following way.

- GF assigns to an object A in $\mathbb{B}$ the object $GFA=G(F(A))$.
- For two objects A,B in $\mathbb{B}$, GF_{AB} is given by $G_{FA,FB}F_{A,B}:\mathbb{B}(A,B)\to\mathbb{D}(GFA,GFB)$.
- For an object A in $\mathbb{B}$, GF^0_A is given by $G(F^0_A)G^0_{FA}:\mathrm{id}_{GFA}\to GF\mathrm{id}_A$.
- For a triple of objects A,B,C in $\mathbb{B}$, the natural isomorphism GF^2_{ABC} has components

$$G(F^2_{g,f})G^2_{Fg,Ff}:GFg\circ GFf\to GF(g\circ f).$$

- It follows that the compatibility diagrams commute.

Where in ordinary category theory we have a natural transformation between functors, in bicategory theory we have strong transformations between pseudofunctors. These are once again not the most general version of a bicategorical natural transformation but they are sufficient for this thesis.

Definition 2.18. (Strong Transformation) Let $F,G:\mathbb{B}\to\mathbb{C}$ be pseudofunctors. A *strong transformation* $\alpha:F\to G$ consists of the following data.

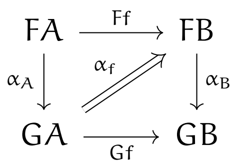
- For each object A in $\mathbb{B}$, a component 1-morphism $\alpha_A:FA\to GA$.
- For each pair of objects A,B in $\mathbb{B}$, a natural isomorphism

$$\alpha:\alpha_A^*G\to(\alpha_B)_*F:\mathbb{B}(A,B)\to\mathbb{C}(FA,GB)$$

  with component 2-isomorphisms

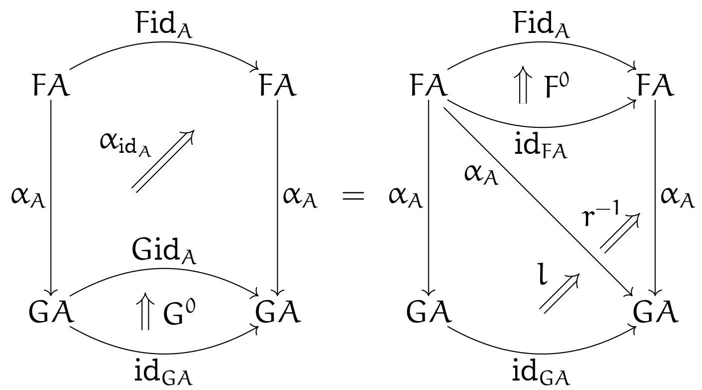
$$\alpha_f:(Gf)\circ\alpha_A\to\alpha_B\circ(Ff).$$

  This can be represented by the following diagram.

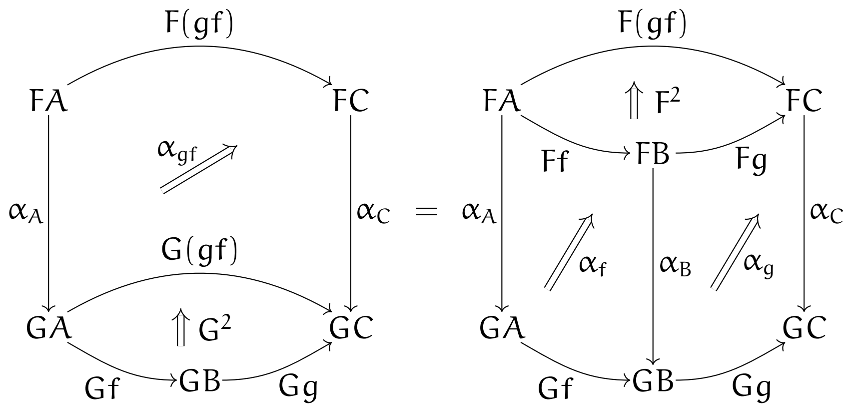


Commutative diagrams like this one where surfaces are equipped with 2-morphisms are explained in great detail in chapter 3 of [JY21]. This data is required to satisfy the following two equalities for all objects A, B, C and 1-morphisms $f: A \to B$, $g: B \to C$.

Unitality:



Naturality:



Example 2.19. (Identity Strong Transformation) Let $F: \mathbb{B} \to \mathbb{C}$ be a pseudofunctor. We can define the identity id_F on F to be the following strong transformation.

- For each object A in $\mathbb{B}$, $\alpha_A : FA \to FA$ is given by id_{FA}.
- For each 1-morphism $f: A \to B$ in $\mathbb{B}$, $\alpha_f : (Ff)\mathrm{id}_{FA} \to \mathrm{id}_{FB}(Ff)$ is given by $l^{-1}{}_{Ff} r_{Ff}$.
- This satisfies the unitality and naturality conditions.

Example 2.20. (Composition) Let $\alpha\colon F \to G$, $\beta\colon G \to H$ be strong transformations for pseudofunctors $F, G, H\colon \mathbb{B} \to \mathbb{C}$. We can define the composite strong transformation $\beta \circ \alpha\colon F \to H$ in the following way.

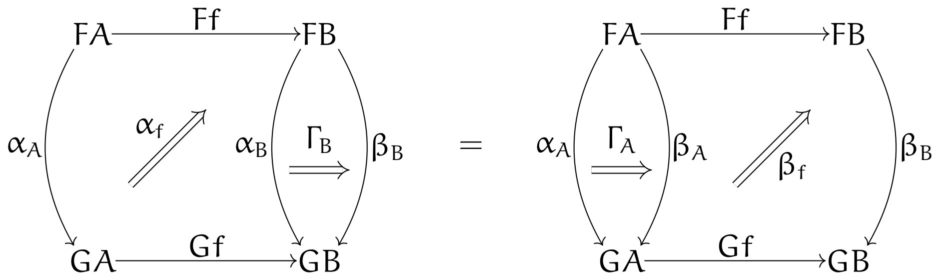
- For each object A in $\mathbb{B}$, we have $(\beta \circ \alpha)_A = \beta_A \circ \alpha_A\colon FA \to HA$.
- For each morphism $f\colon A \to B$, $(\beta \circ \alpha)_f$ is given by

$$(Hf) \circ \beta_A \circ \alpha_A \xrightarrow{\beta_f \circ \mathrm{id}_{\alpha_A}} \beta_B \circ (Gf) \circ \alpha_A \xrightarrow{\mathrm{id}_{\beta_B} \circ \alpha_f} \beta_B \circ \alpha_B \circ (Ff).$$

- This satisfies the unitality and naturality conditions.

Unlike in ordinary category theory, in bicategory theory we further have morphism between strong transformations, since strong transformations carry coherence data in the form of their component 2-isomorphisms.

Definition 2.21. (Modification) Let $\alpha, \beta\colon F \to G$ be strong transformations between pseudofunctors $F, G\colon \mathbb{B} \to \mathbb{C}$. A *modification* $\Gamma\colon \alpha \to \beta$ consists of component 2-morphisms $\Gamma_A\colon \alpha_A \to \beta_A$ in $\mathbb{C}(FA, GA)$ for each object A in $\mathbb{B}$ which satisfy the following equality



for each 1-morphism $f\colon A \to B$ in $\mathbb{B}$.

Example 2.22. (Identity Modification) Let $\alpha\colon F \to G$ be a strong transformation between pseudofunctors $F, G\colon \mathbb{B} \to \mathbb{C}$. The identity modification id_α on α has component 2-morphisms $\mathrm{id}_{\alpha_A}\colon \alpha_A \to \alpha_A$ for each object A in $\mathbb{B}$.

Example 2.23. (Composition) Let $\Gamma\colon \alpha \to \beta$ and $\Lambda\colon \beta \to \gamma$ between strong transformations $\alpha, \beta, \gamma\colon F \to G$ between pseudofunctors $F, G\colon \mathbb{B} \to \mathbb{C}$. The composite $\Lambda\Gamma\colon \alpha \to \gamma$ has component 2-morphism $(\Lambda\Gamma)_A = \Lambda_A \Gamma_A$ for each object A in $\mathbb{B}$.

Akin to how functors and natural transformations assemble into a category, we have the following.

Remark 2.24. For two given pseudofunctors $F,G:\mathbb{B}\to\mathbb{C}$, we have a category $\mathbf{Bicat}(\mathbb{B},\mathbb{C})(F,G)$ which has strong transformations from F to G as objects and modifications between them as morphisms.

Example 2.25. (Whiskering) Let $\Gamma:\alpha\to\beta$ and $\Lambda:\gamma\to\delta$ be modifications between strong transformations $\alpha,\beta:F\to G$, $\gamma,\delta:G\to H$ between pseudofunctors $F,G,H:\mathbb{B}\to\mathbb{C}$. We can define the modification $\Lambda\circ\Gamma:\gamma\circ\alpha\to\delta\circ\beta$ with component 2-morphisms

$$(\Lambda\circ\Gamma)_A=\Lambda_A\circ\Gamma_A:\alpha_A\circ\gamma_A\to\beta_A\circ\delta_A.$$

Remark 2.26. Let $\mathbb{B}$ and $\mathbb{C}$ be bicategories. There exists a bicategory $\mathbf{Bicat}(\mathbb{B},\mathbb{C})$ which has pseudofunctors from $\mathbb{B}$ to $\mathbb{C}$ as objects, strong transformations between these as 1-morphisms and modifications between those as 2-morphisms. The associator and left and right unitors are given component-wise by the associator and left and right unitors in $\mathbb{C}$.

In ordinary category theory we have the notion of an equivalence of categories. If two categories are equivalent, we expect them to look alike under all reasonable aspects that interest us as category theorists. For bicategories we also have a notion of equivalence which we are now able to state.

Definition 2.27. (Equivalence of Bicategories) We call two bicategories $\mathbb{B}$ and $\mathbb{C}$ *equivalent* if there exist pseudofunctors $F:\mathbb{B}\to\mathbb{C}$ and $G:\mathbb{C}\to\mathbb{B}$ such that $GF\simeq \mathrm{id}_{\mathbb{B}}$ in $\mathbf{Bicat}(\mathbb{B},\mathbb{B})$ and $FG\simeq \mathrm{id}_{\mathbb{C}}$ in $\mathbf{Bicat}(\mathbb{C},\mathbb{C})$. We notate this as $\mathbb{B}\simeq\mathbb{C}$ and call F and G *equivalences.* Note that although it carries the same name as an equivalence in a bicategory, this is a weaker notion since the composites need only be equivalent to the identities, not isomorphic.

This definition, while nice to formulate, is often hard to work with. So we can define the following.

Definition 2.28. (Essential Surjectivity) A pseudofunctors $F:\mathbb{B}\to\mathbb{C}$ is called *essentially surjective* if for each object C in $\mathbb{C}$ there exits an object B in $\mathbb{B}$ such that $FB\simeq C$ in $\mathbb{C}$.

Definition 2.29. (Fully Faithfulness) A pseudofunctors $F:\mathbb{B}\to\mathbb{C}$ is called *fully faithful* if for each pair of objects object A,B in $\mathbb{B}$ the functor $F:\mathbb{B}(A,B)\to\mathbb{C}(FA,FB)$ is an equivalence of categories.

Theorem 2.30. *A pseudofunctor $F:\mathbb{B}\to\mathbb{C}$ is an equivalence if and only if it is essentially surjecive and fully faithful.*

A proof of this theorem can be found in chapter 7.4 of [JY21].

Lastly, something that will become interesting to us later when looking at colimits in bicategories, is the notion of adjoint pseudofunctors.

Definition 2.31. (Adjoint Pseudofunctors) Let $\mathbb{B}$ and $\mathbb{C}$ be bicategories. Two pseudofunctors $F:\mathbb{B}\to\mathbb{C}$ and $G:\mathbb{C}\to\mathbb{B}$ are called *adjoint* if there exist strong transformations $\eta:\mathrm{id}_{\mathbb{B}}\to GF$, $\epsilon:FG\to\mathrm{id}_{\mathbb{C}}$ and invertible modifications $\Gamma:\mathrm{id}_F\to(\epsilon F)\circ(F\eta)$ and $\Lambda:(G\epsilon)\circ(\eta G)\to\mathrm{id}_G$. We say that F is *left adjoint* to G, G is *right adjoint* to F and the pseudofunctors F and G form an adjunction.

Proposition 2.32. *Adjoint functors induce an equivalence of categories*

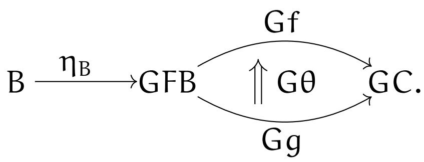
$$\mathbb{C}(FB,C)\simeq\mathbb{B}(B,GC)$$

for each pair of objects B *in* $\mathbb{B}$ *and* C *in* $\mathbb{C}$.

Proof. Let $f:FB\to C$ be a 1-morphism in $\mathbb{B}$. We can define a morphism $f^\flat:B\to GC$ via

$$B\xrightarrow{\eta_B}GFB\xrightarrow{Gf}GC.$$

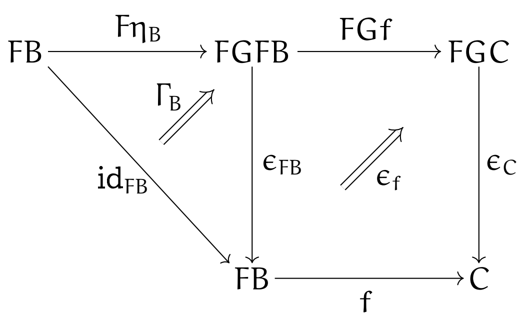
Let $g:FB\to C$ be another 1-morphism and $\theta:f\to g$ a 2-morphism. We can define a 2-morphism $\theta^\flat:f^\flat\to g^\flat$ via



One can check that this defines a functor $(-)^\flat:\mathbb{C}(FB,C)\to\mathbb{B}(B,GC)$. For a 1-moprhism $f:B\to GC$, we can analogously define a morphism $f^\sharp:FB\to C$ via

$$FB\xrightarrow{Ff}FGC\xrightarrow{\epsilon_C}C.$$

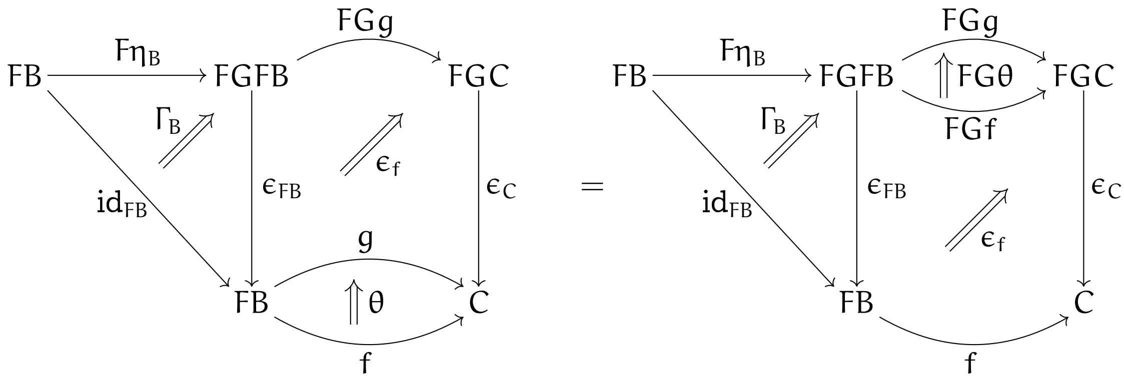
This also defines a functor $(-)^\sharp:\mathbb{B}(B,GC)\to\mathbb{C}(FB,C)$. We will now show that these two functors form an equivalence of categories. We can construct a natural transformation $\mathrm{id}_{\mathbb{C}(FB,C)}\to((-)^\flat)^\sharp$ with components given by the diagram



i.e., we have an isomorphism $f\to(f^\flat)^\sharp$ given by

$$f \xrightarrow{r^{-1}} f \circ \mathrm{id}_{FB} \xrightarrow{f \circ \Gamma_B} f \circ \epsilon_{FB} \circ F\eta_B \xrightarrow{\epsilon_f \circ F\eta_B} \epsilon_C \circ FGf \circ F\eta_B = (f^\flat)^\sharp.$$

That these isomorphism form a natural isomorphism, follows from the equality of the following two diagrams.



We thus have a natural isomorphism $\mathrm{id}_{\mathbb{C}(FB,C)} \cong ((-)^\flat)^\sharp$ and analogously also a natural isomorphism $((-)^\sharp)^\flat \cong \mathrm{id}_{\mathbb{B}(B,GC)}$. Therefore they form an equivalence of categories. $\square$

Example 2.33. Now that we've defined adjoint pseudofunctors, we have the proper langauge to talk about an example of adjoint pseudofunctors that we've already seen. Namely, idempotent completion $\widehat{(-)}: \mathbf{Cat} \to \mathbf{Cat}_{\mathrm{ic}}$ is left adjoint to the forgetful 2-functor $\mathbf{Cat}_{\mathrm{ic}} \to \mathbf{Cat}$. We can now see that indeed defines an adjunction.

2.2 Presheaves and the Yoneda Lemma

With the Yoneda Lemma being so prevalent in category theory that some category theorists claim that more or less everything is a consequence of the Yoneda lemma, it is of no surprise that there also exists a version of the Yoneda lemma for bicategories. In the following we will only state these versions without proving them ourselves. The proofs for these are analogous to their 1-categorical counterparts and can be found in detail in chapter 8 of [JY21].

Definition 2.34. (Bicategory of Presheaves) Let $\mathbb{B}$ be a bicategory. We call a pseudofunctor $\mathbb{B} \to \mathbf{Cat}^{\mathrm{op}}$ a *presheaf* on $\mathbb{B}$. Presheaves, strong transformations and modifications form a bicategory $\mathbf{Bicat}(\mathbb{B}, \mathbf{Cat}^{\mathrm{op}})$. We will denote this bicategory as $\mathbf{Psh}(\mathbb{B})$.

Theorem 2.35. *(Bicategorical Yoneda Lemma) Let $\mathbb{B}$ be a bicategory, A an object in $\mathbb{B}$, and $S: \mathbb{B} \to \mathrm{Cat}^{\mathrm{op}}$ a presheaf on $\mathbb{B}$. The category of strong transformations and modifications $\mathrm{Psh}(\mathbb{B})(\mathbb{B}(-,A), S)$ between the presheaves $\mathbb{B}(-,A)$*

and S *is equivalent to the category* SA *via the functor*

$$\begin{aligned}\mathrm{Psh}(\mathbb{B})(\mathbb{B}(-,A),S) &\to SA\\ \alpha &\mapsto \alpha_A(\mathrm{id}_A).\end{aligned}$$

Furthermore, let $F:\mathbb{B}\to\mathrm{Cat}$ *be a pseudofunctor, the category of strong transformations and and modifications* $\mathrm{Bicat}(\mathbb{B},\mathrm{Cat})(\mathbb{B}(A,-),F)$ *between* $\mathbb{B}(A,-)$ *and* F *is equivalent to the category* FA *via*

$$\begin{aligned}\mathrm{Bicat}(\mathbb{B},\mathrm{Cat})(\mathbb{B}(A,-),F) &\to FA\\ \alpha &\mapsto \alpha_A(\mathrm{id}_A).\end{aligned}$$

Theorem 2.36. *(Bicategorical Yoneda Embedding) Let* $\mathbb{B}$ *be a bicategory. The pseudofunctor*

$$\begin{aligned}\mathcal{Y}:\mathbb{B} &\to \mathrm{Psh}(\mathbb{B})\\ A &\mapsto \mathbb{B}(-,A)\end{aligned}$$

is fully faithful, which means it embeds the bicategory $\mathbb{B}$ *into* $\mathrm{Psh}(\mathbb{B})$.

Corollary 2.37. *Every bicategory* $\mathbb{B}$ *is equivalent to a 2-category.*

Proof. $\mathbf{Psh}(\mathbb{B})$ is a 2-category since $\mathbf{Cat}^{\mathrm{op}}$ is a 2-category. The image of よ now defines a sub-2-category of $\mathbf{Psh}(\mathbb{B})$ equivalent to $\mathbb{B}$. Thus $\mathbb{B}$ is equivalent to a 2-category. □

2.3 Weighted Colimits

We have now laid the necessary groundwork to be able to talk about a bicategorical variant of colimits.

Definition 2.38. (Weighted Colimit) Let $\mathbb{J}$ and $\mathbb{B}$ be bicategories. Given a pseudofunctor $W:\mathbb{J}\to\mathbf{Cat}^{\mathrm{op}}$, which we will call *weight*, and another pseudofunctor $F:\mathbb{J}\to\mathbb{B}$, the *colimit of* F *weighted by* W is given in the following way. We can define a the pseudofunctor

$$\mathbb{B}(F,-):\mathbb{B}\to\mathbf{Bicat}(\mathbb{J},\mathbf{Cat}^{\mathrm{op}}).$$

- It maps an object A in $\mathbb{B}$ onto the pseudofunctor $\mathbb{B}(F-,A):\mathbb{J}\to\mathbf{Cat}^{\mathrm{op}}$.
- For each pair of objects A,B in $\mathbb{B}$, we get a functor

$$\mathbb{B}(F,-):\mathbb{B}(A,B)\to\mathbf{Bicat}(\mathbb{J},\mathbf{Cat}^{\mathrm{op}})(\mathbb{B}(F-,A),\mathbb{B}(F-,B)).$$

  which maps a 1-morphism $f:A\to B$ onto the strong transformation $f_*\mathrm{id}_F$: $\mathbb{B}(F-,A)\to\mathbb{B}(F-,B)$ and a 2-morphism $\alpha:f\to f'$ onto the modification $\alpha\mathrm{id}_{\mathrm{id}_F}$.

We can now form the pseudofunctor

$$\mathbf{Bicat}(\mathbb{J},\mathbf{Cat}^{\mathrm{op}})(W,\mathbb{B}(F,-)):\mathbb{B}\to\mathbf{Cat}.$$

If this pseudofunctor is representable, i.e., there exists an object X and an equivalence of pseudofunctors $\varphi:\mathbb{B}(X,-)\to\mathbf{Bicat}(\mathbb{J},\mathbf{Cat}^{\mathrm{op}})(W,\mathbb{B}(F,-))$, we call X together with φ the *colimit of* F *along* W or just the *weighted colimit of* F and we call φ a *colimiting* strong transformation.

By the Yoneda Lemma, the data of this strong transformation is equivalent to the object $\varphi_X(\mathrm{id}_X)$ in $\mathbf{Bicat}(\mathbb{J},\mathbf{Cat}^{\mathrm{op}})(W,\mathbb{B}(F-,X))$, i.e., a strong transformation $\lambda:W\to\mathbb{B}(F-,X)$ such that precomposition with λ defines an equivalence

$$\lambda^*:\mathbb{B}(X,-)\to\mathbf{Bicat}(\mathbb{J},\mathbf{Cat}^{\mathrm{op}})(W,\mathbb{B}(F,-)).$$

For this thesis we are interested in what it means for a weighted colimit to be preserved by a pseudofunctor and the analogues of absolute colimits and absolute categories.

Definition 2.39. (Preservation of Weighted Colimits) Let $\mathbb{J},\mathbb{B},\mathbb{C}$ be bicategories, $W:\mathbb{J}\to\mathbf{Cat}^{\mathrm{op}}$ a weight and $F:\mathbb{J}\to\mathbb{B}$ and $G:\mathbb{B}\to\mathbb{C}$ pseudofunctors. Let $\mathrm{colim}_W F$ be the weighted colimit of F with strong transformation $\lambda:W\to\mathbb{B}(F-,\mathrm{colim}_W F)$. We say G *preserves* that colimit if precomposition with $G\lambda:W\to\mathbb{C}(GF-,G\mathrm{colim}_W F)$ defines an equivalence

$$(G\lambda)^*:\mathbb{C}(G\mathrm{colim}_W F,-)\to\mathbf{Bicat}(\mathbb{J},\mathbf{Cat}^{\mathrm{op}})(W,\mathbb{C}(GF,-)).$$

Definition 2.40. (Absolute Weighted Colimit) Let $\mathbb{J},\mathbb{B}$ be bicategories, $W:\mathbb{J}\to\mathbf{Cat}^{\mathrm{op}}$ a weight and $F:\mathbb{J}\to\mathbb{B}$ a pseudofunctor. A colimit of F weighted by W is called *absolute* if for every bicategory $\mathbb{C}$ and every pseudofunctor $G:\mathbb{B}\to\mathbb{C}$, it is preserved by G.

Definition 2.41. (Absolute Weight) Let $\mathbb{J}$ be a bicategory. A weight $W:\mathbb{J}\to\mathbf{Cat}^{\mathrm{op}}$ is called *absolute* if for every bicategory $\mathbb{B}$ and every pseudofunctor $F:\mathbb{J}\to\mathbb{B}$, the colimit of F along W is absolute if it exists.

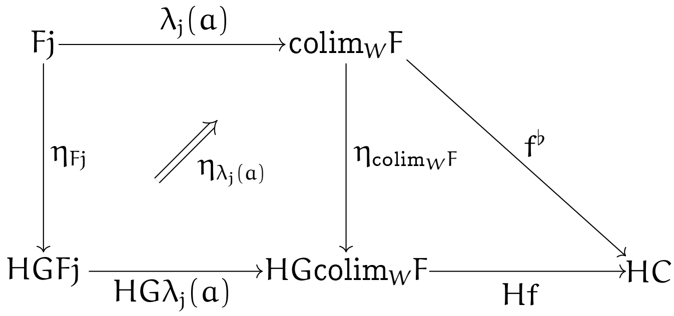
The statement that left adjoint functors preserve colimits also holds true in the bicategorical case.

Proposition 2.42. *Left adjoint pseudofunctors preserve weighted colimits.*

Proof. Let $\mathbb{J},\mathbb{B}$ be bicategories, $W:\mathbb{J}\to\mathbf{Cat}^{\mathrm{op}}$ a weight, $F:\mathbb{J}\to\mathbb{B}$ pseudofunctor and $\lambda:W\to\mathbb{B}(F-,\mathrm{colim}_W F)$ a colimiting strong transformation. Now let $\mathbb{C}$ be another bicategory and $G:\mathbb{B}\to\mathbb{C}$ and $H:\mathbb{C}\to\mathbb{B}$ adjoint pseudofunctors with strong transformations $\eta:\mathrm{id}_{\mathbb{B}}\to HG$ and $\epsilon:GH\to\mathrm{id}_C$. We will show that the strong transformation $G\lambda:W\to\mathbb{C}(GF-,G\mathrm{colim}_W F)$ is colimiting by showing that the following diagram commutes up to isomorphism for all objects C in $\mathbb{C}$

$$\begin{array}{ccc}
\mathbb{C}(\mathrm{G}\,\mathrm{colim}_W\mathrm{F},\mathrm{C}) & \xrightarrow{(\mathrm{G}\lambda)^*} & \mathbf{Bicat}(\mathbb{J},\mathbf{Cat}^{\mathrm{op}})(W,\mathbb{C}(\mathrm{GF}-,\mathrm{C})) \\
\big\downarrow{\scriptstyle (-)^\flat} & & \big\downarrow{\scriptstyle (-)^\flat} \\
\mathbb{B}(\mathrm{colim}_W\mathrm{F},\mathrm{HC}) & \xrightarrow{\lambda^*} & \mathbf{Bicat}(\mathbb{J},\mathbf{Cat}^{\mathrm{op}})(W,\mathbb{B}(\mathrm{F}-,\mathrm{HC}))
\end{array}$$

where $(-)^\flat$ is defined pointwise on strong transformation. All of the functors except $(\mathrm{G}\lambda)^*$ are known to be equivalences of categories, so we only need to show that it commutes up to isomorphism. Let $f\colon \mathrm{G}\,\mathrm{colim}_W\mathrm{F} \to \mathrm{C}$ be a 1-morphism in $\mathbb{C}$. We have an isomorphism $(f_*\mathrm{G}\lambda)^\flat \cong f^\flat_*\lambda$ given by the diagram



where j is an object in $\mathbb{J}$ and a an object in Wj. We can also read off of this diagram that this isomorphism is natural. Thus G preserves the weighted colimit. □

Definition 2.43. (Weighted Colimits in Locally P Bicategories) We can also define the notion of a weighted colimit in a locally P category, where P is a property of categories. Let $\mathbf{Cat}_\mathrm{P}$ be the full subbicategory of $\mathbf{Cat}$ of all categories that have property P and let $\mathbb{B}$ be a locally P bicategory. Let $\mathbb{J}$ be another bicategory and $\mathrm{F}\colon\mathbb{J}\to\mathbb{B}$ a pseudofunctor. A *weight* is now a pseudofunctor $W\colon\mathbb{J}\to\mathbf{Cat}_\mathrm{P}^{\mathrm{op}}$. A *weighted colimit* of F along W now consists of a strong transformation $\lambda\colon W\to \mathbb{B}(\mathrm{F}-,\mathrm{X})$ such that

$$\lambda^*\colon\mathbb{B}(\mathrm{X},\mathrm{A})\to\mathbf{Bicat}(\mathbb{J},\mathbf{Cat}_\mathrm{P}^{\mathrm{op}})(W,\mathbb{B}(\mathrm{F},\mathrm{A}))$$

defines an equivalence of categories for all objects A in $\mathbb{B}$. In general every weighted colimit in a locally P bicategory can be regarded as a weighted colimit in an ordinary bicategory by simply regarding the weight $W\colon\mathbb{J}\to\mathbf{Cat}_\mathrm{P}^{\mathrm{op}}$ as an ordinary weight $W\colon\mathbb{J}\to\mathbf{Cat}^{\mathrm{op}}$. But by restricting ourselves to locally P bicategories and weights that take values in $\mathbf{Cat}_\mathrm{P}$, we will be able to find weights which are absolute and which wouldn't be absolute in the general case. The most relevant case for this thesis will be that of locally idempotent complete bicategories.

2.4 Weighted Colimits in Cat

In many ordinary categories we can give explicit formulas for calculating certain colimits. By explicitly calculating a colimit we usually also gain explicit formulas for the universal property of a colimit which can be used to understand how the colimit interacts with other objects. In the case of the 2-category **Cat**, we also find an explicit formula for a colimit analogous to the one in the 1-category **Set**. This then also allows us to understand some weighted colimits in other categories. By explicitly constructing a weighted colimit for an arbitrary bicategory $\mathbb{J}$, weight $W\colon \mathbb{J} \to \mathbf{Cat}^{\mathrm{op}}$ and pseudofunctor $F\colon \mathbb{J} \to \mathbf{Cat}$, we also show that **Cat** is cocomplete.

Construction 2.44. First we choose a set of objects $\mathrm{Ob}\mathbb{J}$ for $\mathbb{J}$. We look at the diagram

$$\coprod_{j,j',j''\in \mathrm{Ob}\mathbb{J}} Wj'' \times \mathbb{J}(j',j'') \times \mathbb{J}(j,j') \times Fj \overset{d_2^0}{\underset{d_2^2}{\overset{d_2^1}{\rightrightarrows}}} \coprod_{j,j'\in \mathrm{Ob}\mathbb{J}} Wj' \times \mathbb{J}(j,j') \times Fj \overset{d_1^0}{\underset{d_1^1}{\overset{\xleftarrow{s_0^0}}{\rightrightarrows}}} \coprod_{j\in \mathrm{Ob}\mathbb{J}} Wj \times Fj$$

where the maps are defined in the following way

$$\begin{aligned}
d_2^0 &: (a,g,f,x) \mapsto (a,g,F(f)x),\\
d_2^1 &: (a,g,f,x) \mapsto (a,gf,x),\\
d_2^2 &: (a,g,f,x) \mapsto (W(g)a,f,x),\\
d_1^0 &: (a,f,x) \mapsto (a,F(f)x),\\
d_1^1 &: (a,f,x) \mapsto (W(f)a,x),\\
s_0^0 &: (a,x) \mapsto (a,\mathrm{id}_j,x).
\end{aligned}$$

We now define the category $\mathrm{colim}_W F$ by taking the category $\coprod_{j\in \mathrm{Ob}\mathbb{J}} Wj \times Fj$ and freely adding isomorphism determined by d_1^0 and d_1^1 that are subject to relations given by s_0^0, d_2^0, d_2^1 and d_2^2.

For each pair of objects j,j' in $\mathbb{J}$ and an object (a,f,x) in $Wj' \times \mathbb{J}(j,j') \times Fj$ we freely add an isomorphism

$$\gamma_{a,f,x} : (a,F(f)x) \to (W(f)a,x)$$

which assemble into a natural transformation

$$\gamma_{-,-,-} : (-,F(-)-) \to (W(-)-,-) : Wj' \times \mathbb{J}(j,j') \times Fj \to \mathrm{colim}_W F.$$

These natural transformations have to satisfy the following two conditions. For all $j,j',j'' \in \mathrm{Ob}\mathbb{J}$, x in Fj, $f\colon j \to j'$, $g\colon j' \to j''$ and a in Wj'', the diagram

$$\begin{array}{ccccc}
(a,F(g)F(f)x) & \xrightarrow{\gamma_{a,g,F(f)x}} & (W(g)a,F(f)x) & \xrightarrow{\gamma_{W(g)a,f,x}} & (W(f)W(g)a,x) \\
\downarrow{\scriptstyle (\mathrm{id}_a,F^2\mathrm{id}_x)} & & & & \downarrow{\scriptstyle (W^2\mathrm{id}_a,\mathrm{id}_x)} \\
(a,F(gf)x) & & \xrightarrow{\gamma_{a,gf,x}} & & (W(gf)a,x)
\end{array}$$

has to commute and for all $j\in\mathrm{Ob}\mathbb{J}$, x in Fj and a in Wj, the diagram

$$\begin{array}{ccccc}
(a,\mathrm{id}_{Fj}x) & \xrightarrow{=} & (a,x) & \xrightarrow{=} & (\mathrm{id}_{Wj}a,x) \\
\downarrow{\scriptstyle (\mathrm{id}_a,F^0\mathrm{id}_x)} & & & & \downarrow{\scriptstyle (W^0\mathrm{id}_a,\mathrm{id}_x)} \\
(a,F(\mathrm{id}_j)x) & & \xrightarrow{\gamma_{a,\mathrm{id}_j,x}} & & (W(\mathrm{id}_j)a,x)
\end{array}$$

has to commute. We call these two diagrams the naturality and unitality conditions of γ. $\mathrm{colim}_W F$ is now the category obtained by adding these ismorphism subject to the given relations freely to $\coprod_{j\in\mathrm{Ob}\mathbb{J}} Wj\times Fj$. We now also define the functors

$$\gamma_j : W(j)\times F(j)\to \mathrm{colim}_W F$$

which are given by inclusions since $W(j)\times F(j)$ is a subcategory of $\mathrm{colim}_W F$.

We can now define a strong transformation

$$\lambda : W \to \mathbf{Cat}(F-,\mathrm{colim}_W F) : \mathbb{J}\to \mathbf{Cat}^{\mathrm{op}}.$$

For each object j in $\mathbb{J}$, we have a 1-morphism $\lambda_j : Wj\to \mathbf{Cat}(Fj,\mathrm{colim}_W F)$ given by $\lambda_j(a)(x)=\gamma_j(a,x)=(a,x)$ for each a in Wj and x in Fj. Since γ_j is a functor, both λ_j and $\lambda_j(a)$ also define functors.

For each pair of objects j,j' in $\mathbb{J}$, we also have a natural transformation

$$\lambda : \lambda_{j'}^{*}\mathbf{Cat}(F-,\mathrm{colim}_W F)\to(\lambda_j)_* W : \mathbb{J}(j,j')\to \mathbf{Cat}(Wj',\mathbf{Cat}(Fj,\mathrm{colim}_W F))$$

with component morphisms $\lambda_f : \mathbf{Cat}(Ff,\mathrm{colim}_W F)\lambda_{j'}\to\lambda_j Wf$ which are defined by

$$\begin{aligned}\lambda_{f,a,x}=\gamma_{a,f,x} : (\mathbf{Cat}(Ff,\mathrm{colim}_W F)\lambda_{j'}(a))(x) &= \lambda_{j'}(a)(F(f)x) \\ &= (a,F(f)x)\to(W(f)a,x)=\lambda_j(W(f)a)(x)\end{aligned}$$

for each a in Wj' and x in Fj. The naturality of λ and λ_f and $\lambda_{f,a}$ all follow from the naturality of γ. Lastly we need to check that λ satisfies the naturality and unitality conditions, this follows since γ needs to satisfy its own naturality and unitality conditions.

Theorem 2.45. *Given a bicategory $\mathbb{J}$, a weight $W:\mathbb{J}\to\mathrm{Cat}^{\mathrm{op}}$ and a pseudofunctor $F:\mathbb{J}\to\mathrm{Cat}$, construction 2.44 defines a weighted colimit of F along W.*

Proof. We next need to check that precomposition with $\lambda : W \to \mathbf{Cat}(F-, \operatorname{colim}_W F)$ defines an equivalence of pseudofunctors

$$\mathbf{Cat}(\operatorname{colim}_W F, -) \to \mathbf{Bicat}(\mathbb{J}, \mathbf{Cat}^{\mathrm{op}})(W, \mathbf{Cat}(F, -)).$$

For this it suffices to check that it defines an equivalence of categories

$$\mathbf{Cat}(\operatorname{colim}_W F, \mathcal{C}) \to \mathbf{Bicat}(\mathbb{J}, \mathbf{Cat}^{\mathrm{op}})(W, \mathbf{Cat}(F-, \mathcal{C}))$$

for each category $\mathcal{C}$. As it turns out, a functor $\operatorname{colim}_W F \to \mathcal{C}$ is determined by the same data as a strong transformation $W \to \mathbf{Cat}(F-, \mathcal{C})$ just presented in different ways. This will lead to an isomorphism of categories.

A strong transformation $\epsilon : W \to \mathbf{Cat}(F-, \mathcal{C})$ is given by a $\mathbb{J}$-indexed family of functors $\epsilon_j : Wj \to \mathbf{Cat}(Fj, \mathcal{C})$ and for each morphism $f : j \to j'$ in $\mathbb{J}$ a natural isomorphism $\epsilon_f : \mathbf{Cat}(Ff, \mathcal{C})\epsilon_{j'} \to \epsilon_j Wf : Wj' \to \mathbf{Cat}(Fj, \mathcal{C})$ such that the ϵ_f are natural in f and satisfy the naturality and unitality conditions.

Using the fact that for categories $\mathcal{C}_1, \mathcal{C}_2, \mathcal{D}$ we have an isomorphism of categories

$$\mathbf{Cat}(\mathcal{C}_1 \times \mathcal{C}_2, D) \cong \mathbf{Cat}(\mathcal{C}_1, \mathbf{Cat}(\mathcal{C}_2, \mathcal{D}))$$

given by currying, know that the family $(\epsilon_j)_{j \in \mathrm{Ob}\mathbb{J}}$ is equivalent to a $\mathbb{J}$-indexed family of functors $\widetilde{\epsilon}_j : Wj \times Fj \to \mathcal{C}$ and the ϵ_f turn into natural transformations

$$\widetilde{\epsilon}_f : \widetilde{\epsilon}_{j'}(-, F(f)-) \to \widetilde{\epsilon}_j(W(f)-, -) : Wj' \times Fj \to \mathcal{C}$$

Naturality in f now means that they assemble into a natural transformation

$$\widetilde{\epsilon} : \widetilde{\epsilon}_{j'}(-, F(-)-) \to \widetilde{\epsilon}_j(W(-)-, -) : Wj' \times \mathbb{J}(j, j') \times Fj \to \mathcal{C}.$$

The naturality condition tranlates into the following commutative diagram

$$\begin{array}{ccccc}
\widetilde{\epsilon}_{j''}(a, F(g)F(f)x) & \xrightarrow{\widetilde{\epsilon}_{a,g,F(f)x}} & \widetilde{\epsilon}_{j'}(W(g)a, F(f)x) & \xrightarrow{\widetilde{\epsilon}_{W(g)a,f,x}} & \widetilde{\epsilon}_j(W(f)W(g)a, x) \\
\downarrow{\scriptstyle \widetilde{\epsilon}_{j''}(\mathrm{id}_a, F^2\mathrm{id}_x)} & & & & \downarrow{\scriptstyle \widetilde{\epsilon}_j(W^2\mathrm{id}_a, \mathrm{id}_x)} \\
\widetilde{\epsilon}_{j''}(a, F(gf)x) & & \xrightarrow{\qquad\widetilde{\epsilon}_{a,gf,x}\qquad} & & \widetilde{\epsilon}_j(W(gf)a, x)
\end{array}$$

which has to hold for all objects j, j', j'' in $\mathbb{J}$, morphisms $f : j \to j'$ and $g : j' \to j''$ and objects a in Wj'' and x in Fj. The unitality condition tranlates into the diagram

$$\begin{array}{ccccc}
\widetilde{\epsilon}_j(a, \mathrm{id}_{Fj}x) & \xrightarrow{=} & \widetilde{\epsilon}_j(a, x) & \xrightarrow{=} & \widetilde{\epsilon}_j(\mathrm{id}_{Wj}a, x) \\
\downarrow{\scriptstyle \widetilde{\epsilon}_j(\mathrm{id}_a, F^0\mathrm{id}_x)} & & & & \downarrow{\scriptstyle \widetilde{\epsilon}_j(W^0\mathrm{id}_a, \mathrm{id}_x)} \\
\widetilde{\epsilon}_j(a, F(\mathrm{id}_j)x) & & \xrightarrow{\qquad\widetilde{\epsilon}_{a,\mathrm{id}_j,x}\qquad} & & \widetilde{\epsilon}_j(W(\mathrm{id}_j)a, x)
\end{array}$$

which has to commute for all objects j in $\mathbb{J}$, a in Wj and x in Fj. We can already see that these are exactly the conditions we asked of γ when constructing $\mathrm{colim}_W F$. We will now look at the data that a functor $G\colon \mathrm{colim}_W F \to \mathcal{C}$ consists of. Since $\mathrm{colim}_W F$ was constructed by adding isomorphisms to the category $\coprod_{j \in \mathrm{Ob}\mathbb{J}} Wj \times Fj$, the functor G consists a family of functors $G_j\colon Wj \times Fj \to \mathcal{C}$ that also has to map the components γ on isomorphisms in $\mathcal{C}$ that have to satisfy the same properties as γ. So we have a natural transformation

$$G(\gamma_{-,-,-})\colon G(-,F(-)-) \to G(W(-)-,-)\colon Wj' \times \mathbb{J}(j,j') \times Fj \to \mathcal{C}$$

and the components of this natural transformation also have to satisfy the unitality and naturality conditions. So we indeed see that a functor $\mathrm{colim}_W F \to \mathcal{C}$ consists of the same data as a strong transformation $W \to \mathbf{Cat}(F-,\mathcal{C})$ and the presentation of this data only differs by currying. This currying is also induced by precomposition with λ since given a functor $G\colon \mathrm{colim}_W F \to \mathcal{C}$, the strong transformation $G_*\lambda\colon W \to \mathbf{Cat}(F-,\mathcal{C})$ is defined by

$$(G_*\lambda)_j(a)(x) = (G_*\lambda_j)(a)(x) = (G\lambda_j(a))(x) = G(\lambda_j(a)(x)) = G(a,x)$$

for objects j in $\mathbb{J}$, a in Wj and x in Fj, and by

$$(G_*\lambda)_{f,a,x} = G\lambda_{f,a,x} = G\gamma_{a,f,x}$$

for objects j,j' in $\mathbb{J}$, a morphism $f\colon j \to j'$ and objects a in Wj' and x in Fj. To show that this induces an isomorphism of categories

$$\mathbf{Cat}(\mathrm{colim}_W F,\mathcal{C}) \cong \mathbf{Bicat}(\mathbb{J},\mathbf{Cat})(W,\mathbf{Cat}(F-,\mathcal{C}))$$

we also have to show that a natural transformation between such functors also only differs by currying from a modification between such strong transformations. Let $\epsilon,\eta\colon W \to \mathbf{Cat}(F-,\mathcal{C})$ be two strong transformations and let $\Gamma\colon \epsilon \to \eta$ be a modification between them. Γ consists of a $\mathbb{J}$-indexed family of natural transformations

$$\Gamma_j\colon \epsilon_j \to \eta_j\colon Wj \to \mathbf{Cat}(Fj,\mathcal{C})$$

which since Γ is a modification has to satisfy a certain property. Using currying, this the natural transformations Γ_j correspond to natural transformations

$$\widetilde{\Gamma}_j\colon \widetilde{\epsilon} \to \widetilde{\eta}\colon Wj \times Fj \to \mathcal{C}$$

and the property they need to satisfy can be written as the diagram

$$\begin{array}{ccc} \widetilde{\epsilon}_{j'}(a,F(f)x) & \xrightarrow{\widetilde{\epsilon}_{a,f,x}} & \widetilde{\epsilon}_j(W(f)a,x) \\ \downarrow{\scriptstyle \widetilde{\Gamma}_{j',a,F(f)x}} & & \downarrow{\scriptstyle \widetilde{\Gamma}_{j,W(f)a,x}} \\ \widetilde{\eta}_{j'}(a,F(f)x) & \xrightarrow{\widetilde{\eta}_{a,f,x}} & \widetilde{\eta}_j(W(f)a,x) \end{array}$$

which has to commute for all objects j,j' in $\mathbb{J}$, morphisms $f:j\to j'$ and objects a in Wj' and x in Fj. Now let $G,H:\mathrm{colim}_W F\to\mathcal{C}$ be two functors and let $\varphi:G\to H$ be a natural transformations. We've already seen that due to the structure of $\mathrm{colim}_W F$ the two functors G,H consists of $\mathbb{J}$-indexed families of functors $G_j,H_j:Wj\times Fj\to\mathcal{C}$ together with the natural transformations $G(\gamma)$ and $H(\gamma)$. A natural transformation $\varphi:G\to H$ now consist of a $\mathbb{J}$-indexed family of natural transformations $\varphi_j:G_j\to H_j$ which also have to be natural with respect to γ. This can be represented by the diagram

$$\begin{array}{ccc} G_{j'}(a,F(f)x) & \xrightarrow{G(\gamma_{a,f,x})} & G_j(W(f)a,x) \\ \downarrow{\scriptstyle \varphi_{j',(a,F(f)x)}} & & \downarrow{\scriptstyle \varphi_{j,(W(f)a,x)}} \\ H_{j'}(a,F(f)x) & \xrightarrow{H(\gamma_{a,f,x})} & H_j(W(f)a,x) \end{array}$$

which has to commute for all objects j,j' in $\mathbb{J}$, morphisms $f:j\to j'$ and objects a in Wj' and x in Fj. So now we also see that a natural transformation $\varphi:G\to H$ between functors $G,H:\mathrm{colim}_W F\to\mathcal{C}$ consists of the same data as a modification $\Gamma:\epsilon\to\eta$ between strong transformations $\epsilon,\eta:W\to\mathbf{Cat}(F-,\mathcal{C})$ and the presentation of this data also only differs by currying.

We have therefore shown that precomposition with $\lambda:W\to\mathbf{Cat}(F-,\mathrm{colim}_W F)$, which does the same as currying, actually defines an isomorphism

$$\lambda^*:\mathbf{Cat}(\mathrm{colim}_W F,\mathcal{C})\to\mathbf{Bicat}(\mathbb{J},\mathbf{Cat})(W,\mathbf{Cat}(F-,\mathcal{C})).$$

for each category $\mathcal{C}$. Thus $\mathrm{colim}_W F$ together with λ forms a colimit of F along W. □

Corollary 2.46. *The bicategory* Cat *of categories is cocomplete, i.e., for every bicategory* $\mathbb{J}$*, every weight* $W:\mathbb{J}\to\mathrm{Cat}^{\mathrm{op}}$ *and every pseudofunctor* $F:\mathbb{J}\to\mathrm{Cat}$ *the colimit of* F *weighted by* W *exists.*

We also want to show that $\mathbf{Cat}_{\mathrm{ic}}$ is cocomplete as a locally idempotent complete bicategory. For this we will first have to show that it actually is locally idempotent complete.

Lemma 2.47. *For a category* $\mathcal{C}$ *and an idempotent complete category* $\mathcal{D}$*, the category* $\mathrm{Cat}(\mathcal{C},\mathcal{D})$ *of functors and natural transformations is idempotent complete.*

Proof. Let $F:\mathcal{C}\to\mathcal{D}$ be a functor and $p:F\to F$ an idempotent natural transformation, i.e., for every object C in $\mathcal{C}$ the morphism $p_C:FC\to FC$ is an idempotent. Since these are morphisms in $\mathcal{D}$, we can choose a splitting for every idempotent p_C. We now choose for every C in $\mathcal{C}$ an object SC in $\mathcal{D}$ and morphisms $f_C:FC\to SC$

and $g_C : SC \to FC$ such that $g_C f_C = p_C$ and $f_C g_C = \mathrm{id}_{SC}$. We can now turn S into a functor. Let $h : C \to D$ be a morphism in $\mathcal{C}$. We define $S(h) = f_D F(h) g_C$. We now have $S(\mathrm{id}_C) = f_C F(\mathrm{id}_C) g_C = f_C g_C = \mathrm{id}_{SC}$ and for another morphism $k : D \to E$, we have

$$\begin{aligned} S(k)S(h) &= f_E F(k) g_D f_D F(h) g_C = f_E F(k) p_D F(h) g_C = f_E F(k) F(h) p_C g_C \\ &= f_E F(kh) g_C f_C g_C = f_E F(kh) g_C \mathrm{id}_{SC} = S(kh) \end{aligned}$$

which shows that S defines a functor. We lastly need to show that the f_C and g_C define natural transformations $f : F \to S$ and $g : S \to F$. For this we need to check that the following two diagrams commute.

$$\begin{array}{ccc} FC & \xrightarrow{F(h)} & FD \\ \downarrow{\scriptstyle f_C} & & \downarrow{\scriptstyle f_D} \\ SC & \xrightarrow{S(h)} & SD \end{array} \qquad \begin{array}{ccc} SC & \xrightarrow{S(h)} & SD \\ \downarrow{\scriptstyle g_C} & & \downarrow{\scriptstyle g_D} \\ FC & \xrightarrow{F(h)} & FD \end{array}$$

We can show

$$S(h) f_C = f_D F(h) g_C f_C = f_D F(h) p_C = f_D p_D F(h) = f_D g_D f_D F(h) = \mathrm{id}_{SD} f_D F(h) = f_D F(h)$$

and thus $f : F \to S$ defines a natural transformation and

$$g_D S(h) = g_D f_D F(h) g_C = p_D F(h) g_C = F(h) p_C g_C = F(h) g_C f_C g_C = F(h) g_C \mathrm{id}_{SC} = F(h) g_C$$

and thus $g : S \to F$ defines a natural transformation. The natural transformations f and g now satisfy $gf = p$ and $fg = \mathrm{id}_S$ and thus p splits. □

Corollary 2.48. *The bicategories* $\mathrm{Cat}_{\mathrm{ic}}$ *and* $\mathrm{Cat}_{\mathrm{ic}}^{\mathrm{op}}$ *are locally idempotent complete.*

Theorem 2.49. *The locally idempotent complete bicategory* $\mathrm{Cat}_{\mathrm{ic}}$ *of idempotent complete categories is cocomplete, i.e., for every bicategory* $\mathbb{J}$, *every weight* $W : \mathbb{J} \to \mathrm{Cat}_{\mathrm{ic}}^{\mathrm{op}}$ *and every pseudofunctor* $F : \mathbb{J} \to \mathrm{Cat}_{\mathrm{ic}}$ *the colimit of* F *weighted by* W *exists.*

Proof. Let $\mathbb{J}$ be a bicategory, $W : \mathbb{J} \to \mathbf{Cat}_{\mathrm{ic}}^{\mathrm{op}}$ a weight and $F : \mathbb{J} \to \mathbf{Cat}_{\mathrm{ic}}$ a pseudofunctor. We can regard both W and F as pseudofunctors taking values in **Cat**. By the previous theorem we now know that we can construct a category $\mathcal{C}$ along with a strong transformation $\lambda : W \to \mathbf{Cat}(F-, \mathcal{C})$ such that we have a natural equivalence

$$\lambda^* : \mathbf{Cat}(\mathcal{C}, -) \to \mathbf{Bicat}(\mathbb{J}, \mathbf{Cat}^{\mathrm{op}})(W, \mathbf{Cat}(F, -)).$$

Since we have a left adjoint pseudofunctor $\widehat{(-)} : \mathbf{Cat} \to \mathbf{Cat}_{\mathrm{ic}}$, we have a natural equivalence

$$(\widehat{(-)}\lambda)^* : \mathbf{Cat}_{\mathrm{ic}}(\widehat{\mathcal{C}}, -) \to \mathbf{Bicat}(\mathbb{J}, \mathbf{Cat}^{\mathrm{op}})(W, \mathbf{Cat}_{\mathrm{ic}}(\widehat{(-)}F, -))$$

where $\widehat{(-)}\lambda$ is defined by

$$(\widehat{(-)}\lambda)_j(a) = \widehat{\lambda_j(a)} : \widehat{Fj} \to \widehat{\mathcal{C}}$$

for each j in $\mathbb{J}$ and a in Wj. Since F already takes values in $\mathbf{Cat}_{\mathrm{ic}}$, F and $\widehat{(-)}F$ are equivalent with an equivalence given by $\iota_F : F \to \widehat{(-)}F$ which has components given by $\iota_{Fj} : Fj \to \widehat{Fj}$. We also know that the diagram

$$\begin{array}{ccc} \widehat{Fj} & \xrightarrow{\widehat{\lambda_j(a)}} & \widehat{\mathcal{C}} \\ {\scriptstyle \iota_{Fj}}\uparrow & & \uparrow{\scriptstyle \iota_{\mathcal{C}}} \\ Fj & \xrightarrow{\lambda_j(a)} & \mathcal{C} \end{array}$$

commutes. These two facts combined tell us that we have an equivalence

$$((\iota_{\mathcal{C}})_*\lambda)^* : \mathbf{Cat}_{\mathrm{ic}}(\widehat{\mathcal{C}}, -) \to \mathbf{Bicat}(\mathbb{J}, \mathbf{Cat}^{\mathrm{op}})(W, \mathbf{Cat}_{\mathrm{ic}}(F, -))$$

and since W takes values in $\mathbf{Cat}^{\mathrm{op}}_{\mathrm{ic}}$ and $\mathbf{Cat}_{\mathrm{ic}}$ is locally idempotent complete, we get the desired result that we have an equivalence

$$((\iota_{\mathcal{C}})_*\lambda)^* : \mathbf{Cat}_{\mathrm{ic}}(\widehat{\mathcal{C}}, -) \to \mathbf{Bicat}(\mathbb{J}, \mathbf{Cat}^{\mathrm{op}}_{\mathrm{ic}})(W, \mathbf{Cat}_{\mathrm{ic}}(F, -))$$

□

Another bicategory that interest us is the bicategory of presheaves over a given bicategory.

Theorem 2.50. *Let $\mathbb{B}$ be a bicategory. The bicategory* Psh($\mathbb{B}$) *is cocomplete.*

Proof. Let $\mathbb{J}$ be a bicategory, $W : \mathbb{J} \to \mathbf{Cat}^{\mathrm{op}}$ a weight and $F : \mathbb{J} \to \mathbf{Psh}(\mathbb{B})$ a pseudofunctor. We can construct a weighted colimit of F along W in the following way. Let A be an object in $\mathbb{B}$. We now have the pseudofunctor $F(-)(A) : \mathbb{J} \to \mathbf{Cat}$ and since $\mathbf{Cat}$ is cocomplete the weighted colimit of $F(-)(A)$ along W exists, i.e., we have a colimiting strong transformation $\lambda_A : W \to \mathbf{Cat}(F(-)(A), \mathrm{colim}_W F(-)(A))$.

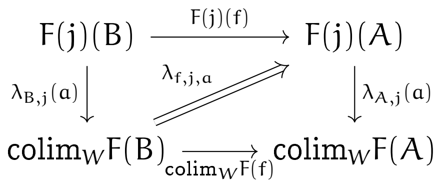
We now define the presheaf $\mathrm{colim}_W F : \mathbb{B} \to \mathbf{Cat}^{\mathrm{op}}$ via $\mathrm{colim}_W F(A) = \mathrm{colim}_W F(-)(A)$. Let B be another object in $\mathbb{B}$ and $f : A \to B$ a morphism. We can now define the strong transformation $F(-)(f)^*\lambda_A : W \to \mathbf{Cat}(F(-)(B), \mathrm{colim}_W F(A))$ which has components $(F(-)(f)^*\lambda_A)_j(a) = \lambda_{A,j}(a) F(j)(f)$

$$F(j)(B) \xrightarrow{F(j)(f)} F(j)(A) \xrightarrow{\lambda_{A,j}(a)} \mathrm{colim}_W F(A)$$

for each j in $\mathbb{J}$ and a in Wj. By the universal property of the colimit, we get a morphism

$$\mathrm{colim}_W F(f) : \mathrm{colim}_W F(B) \to \mathrm{colim}_W F(A)$$

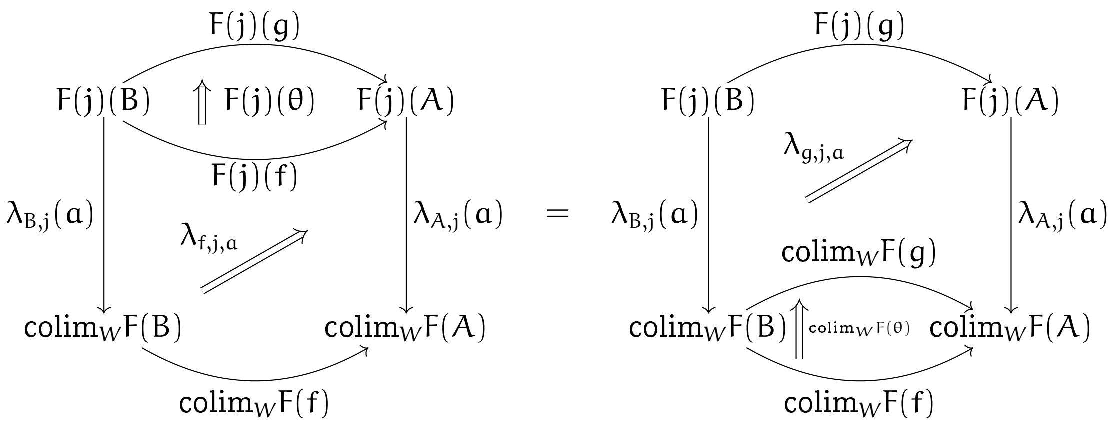
and an isomorphism $\lambda_f : \mathrm{colim}_W F(f)_* \lambda_B \to F(-)(f)^* \lambda_A$ which we can represent by the following diagram



For a second 1-morphism $g : A \to B$ and a 2-morphism $\theta : f \to g$, we get a morphism of strong transformations

$$F(-)(\theta)^* \lambda_A : F(-)(f)^* \lambda_A \to F(-)(g)^* \lambda_A$$

which gives us a morphism $\mathrm{colim}_W F(\theta) : \mathrm{colim}_W F(f) \to \mathrm{colim}_W F(g)$. This can be represented as

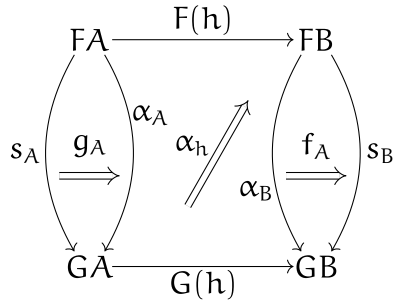


The presheaf $\mathrm{colim}_W F$ is now well defined along with a strong transformation $\lambda : W \to \mathbf{Psh}(\mathbb{B})(F, \mathrm{colim}_W F)$ with components $\lambda_{j,A}(a) = \lambda_{A,j}(a)$. We must now show that λ defines a colimiting strong transformation. Let $\kappa : W \to \mathbf{Psh}(\mathbb{B})(F, S)$ be another strong transformation for a presheaf S. For an object A in $\mathbb{B}$, we can then define the natural transformation $\kappa_A : \mathbf{Psh}(\mathbb{B})(F(-)(A), S(A))$ and by the universal property of the colimit, we get a morphism $\varphi_A : \mathrm{colim}_W F(A) \to S(A)$ such that $\kappa_A \cong (\varphi_A)_* \lambda_A$. The φ_A assemble into a morphism of presheafs $\varphi : \mathrm{colim}_W F \to S$ and we get an isomorphism $\kappa \cong \varphi_* \lambda$. This shows that λ defines a colimit. □

Lastly, we also want to show that the bicategory $\mathbf{Psh}_{\mathrm{ic}}(\mathbb{B})$ of presheaves taking values in idempotent complete categories is cocomplete as a locally idempotent complete bicategory.

Lemma 2.51. *For a bicategory* $\mathbb{B}$ *and a locally idempotent complete bicategory* $\mathbb{C}$ *the bicategory* $\mathrm{Bicat}(\mathbb{B},\mathbb{C})$ *is locally idempotent complete.*

Proof. Let $F,G:\mathbb{B}\to\mathbb{C}$ be pseudofuncors, $\alpha:F\to G$ a strong transformation and $p:\alpha\to\alpha$ an idempotent modification, i.e., for each object A in $\mathbb{B}$ the morphism $p_A:\alpha_A\to\alpha_A$ in $\mathbb{C}(FA,GA)$ is idempotent. Since $\mathbb{C}$ is locally idempotent complete, we can choose a splitting for each p_A for all A in $\mathbb{B}$. We therefore have 1-morphisms $s_A: FA\to GA$, and 2-morphisms $f_A:\alpha_A\to s_A$ and $g_A:s_A\to\alpha_A$ such that $g_Af_A=p_A$ and $f_Ag_A=\mathrm{id}_{s_A}$. $s:F\to G$ forms a strong transformation with component 2-morphisms $s_h:G(h)s_A\to s_BF(h)$ given by



for a 1-morphism $h:A\to B$ in $\mathbb{B}$. $f:\alpha\to s$ and $g:s\to\alpha$ now form modifications such that $gf=p$ and $fg=\mathrm{id}_s$ and thus p splits. ☐

Corollary 2.52. *Let* $\mathbb{B}$ *be a bicategory. The bicategory* $\mathrm{Psh}_{\mathrm{ic}}(\mathbb{B})=\mathrm{Bicat}(\mathbb{B},\mathrm{Cat}_{\mathrm{ic}}^{\mathrm{op}})$ *of idempotent complete presheafs on* $\mathbb{B}$ *is locally idempotent complete.*

Combining the last two theorems and their proofs also yields the following result.

Corollary 2.53. *Let* $\mathbb{B}$ *be a bicategory. The locally idempotent complete bicategory* $\mathrm{Psh}_{\mathrm{ic}}(\mathbb{B})$ *is cocomplete.*

3 Karoubi Completion

Having now laid the bicategorical prerequisites, we can continue to look at the bicategorical analogue of idempotent completion. From now on, we will suppress coherence data whenever appropriate. Formally, this can be justified since every bicategory is equivalent to a 2-category.

3.1 2-Idempotents and their Splittings

We will now go on to define the bicategorical analogue of an idempotent and of its splitting. These definitions follow [GJF19].

Definition 3.1. (2-Idempotent) Let $\mathbb{B}$ be a bicategory. A *2-idempotent* in $\mathbb{B}$ consists of an object A in $\mathbb{B}$, a 1-morphism $p\colon A\to A$ and 2-morphisms $m\colon p^2\to p$ and $\Delta\colon p\to p^2$ such that

$$(\mathrm{id}_p\circ m)\cdot(\Delta\circ \mathrm{id}_p)=(m\circ \mathrm{id}_p)\cdot(\mathrm{id}_p\circ\Delta)=\Delta\cdot m \text{ and}$$
$$m\cdot\Delta=\mathrm{id}_p.$$

We denote a 2-idempotent by (A,p,m,Δ) and whenever it is clear from context, we will simply denote it as A_p.

Definition 3.2. (2-Idempotent Splitting) Let $\mathbb{B}$ be a bicategory and let (A,p,m,Δ) be a 2-idempotent in $\mathbb{B}$. A *splitting* of A_p is given by an object B in $\mathbb{B}$, 1-morphisms $f\colon A\to B$ and $g\colon B\to A$, 2-morphisms $\varphi\colon f\circ g\to \mathrm{id}_B$, $\psi\colon \mathrm{id}_B\to f\circ g$ and an isomorphism $\gamma\colon g\circ f\to p$ such that $\varphi\cdot\psi=\mathrm{id}_{\mathrm{id}_B}$, $m=\gamma\cdot(\mathrm{id}_g\circ\varphi\circ\mathrm{id}_f)\cdot(\gamma^{-1}\circ\gamma^{-1})$ and $\Delta=(\gamma\circ\gamma)\cdot(\mathrm{id}_g\circ\psi\circ\mathrm{id}_f)\cdot\gamma^{-1}$. We say that the 2-idempotent p *splits* and we call B a *retract* of A. We will later show that in certain cases a splitting of a 2-idempotent is unique up to equivalence.

Definition 3.3. (Idempotent Completeness) We call a bicategory $\mathbb{B}$ *2-idempotent complete* if it is locally idempotent complete and if every 2-idempotent $\mathbb{B}$ splits.

Definition 3.4. (Free Walking 2-Idempotent (Splitting)) We will define the bicategory $\spadesuit_2$ to be the bicategory with two objects X and Y, 1-morphisms freely generated by $f\colon X\to Y$ and $g\colon Y\to X$ and 2-morphisms freely generated by $\varphi\colon f\circ g\to \mathrm{id}_Y$ and $\psi\colon \mathrm{id}_Y\to f\circ g$ satisfying the relation $\varphi\cdot\psi=\mathrm{id}_{\mathrm{id}_Y}$. We will call this bicategory $\spadesuit_2$ the *free walking 2-idempotent splitting*.

The bicategory $\clubsuit_2$ is defined to be the full subbicategory of $\spadesuit_2$ on the object X. We will call this bicategory the *free walking 2-idempotent*. We then of course have a fully faithful pseudofunctor $\iota\colon\clubsuit_2\to\spadesuit_2$.

Remark 3.5. We can now also think of a 2-idempotent in $\mathbb{B}$ as a pseudofunctor $F\colon\clubsuit_2\to\mathbb{B}$ since every such functor determines a 2-idempotent and we can define such a functor simply by choosing a 2-idempotent in $\mathbb{B}$. Furthermore, a splitting of F is then given by a functor $F'\colon\spadesuit_2\to\mathbb{B}$ such that $F'\iota\simeq F$.

3.2 Karoubi Completion of a Bicategory

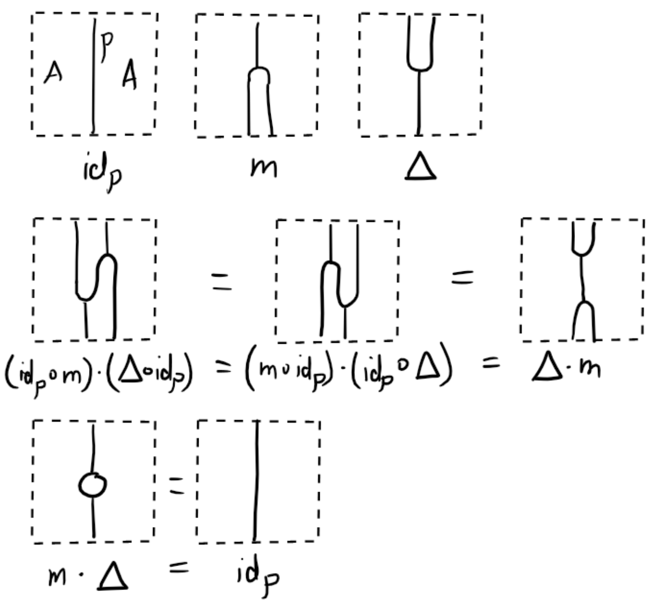
In the following chapter, we define the Karoubi completion of a locally idempotent complete bicategory $\mathbb{B}$ and show that it forms an idempotent complete bicategory and furthermore that it is universal among all bicategories with this property.

Definition 3.6. (Karoubi completion) Let $\mathbb{B}$ be a locally idempotent complete bicategory. We define the *Karoubi completion* $\widehat{\mathbb{B}}$ to have the following data.

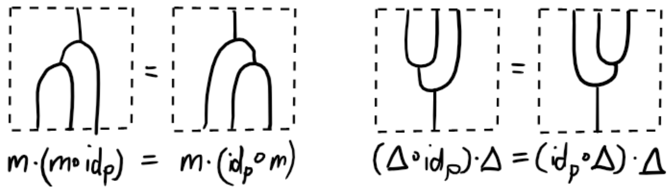
- Objects in $\widehat{\mathbb{B}}$ are 2-idempotents in $\mathbb{B}$, i.e., an object in $\widehat{\mathbb{B}}$ is a collection (A,p,m,Δ) consisting of an object A in $\mathbb{B}$, a 1-morphism $p\colon A\to A$ in $\mathbb{B}$ and 2-morphisms $m\colon p^2\to p$ and $\Delta\colon p\to p^2$ in $\mathbb{B}$ such that

$$(\mathrm{id}_p\circ m)\cdot(\Delta\circ\mathrm{id}_p)=(m\circ\mathrm{id}_p)\cdot(\mathrm{id}_p\circ\Delta)=\Delta\cdot m \text{ and}$$
$$m\cdot\Delta=\mathrm{id}_p.$$

Graphically, this can be represented as follows.



Two other useful properties follow from these requirements, namely associativity and coassociativity which graphically can be written in the following way.

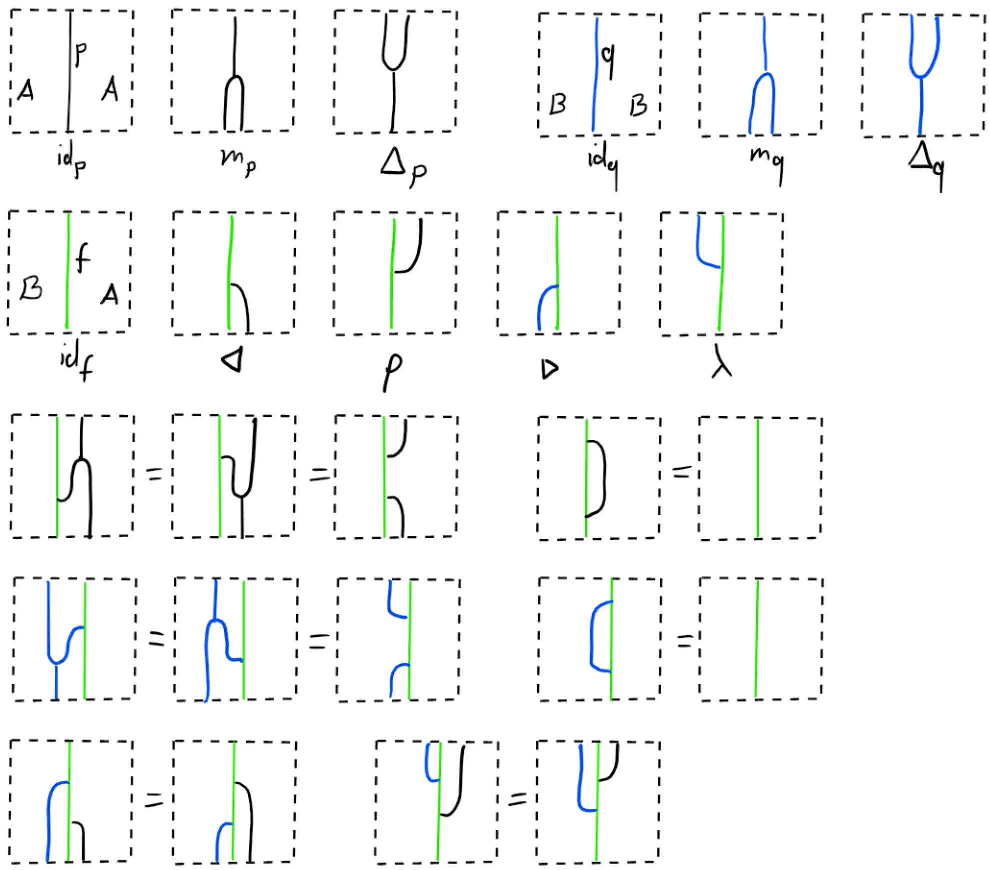


With this, (A,p,m,Δ) forms a special non-unital non-counital Frobenius algebra. Whenever it is clear from context, we will denote (A,p,m,Δ) as A_p.

- For each pair of objects (A,p,m_p,Δ_p) and (B,q,m_q,Δ_q) in $\widehat{\mathbb{B}}$, we have a Hom-category $\widehat{\mathbb{B}}(A_p,B_q)$. A 1-morphism in $\widehat{\mathbb{B}}$ is a collection $(f,\triangleleft,\rho,\triangleright,\lambda)$ consisting of a 1-morphism $f:A\to B$ in $\mathbb{B}$, and 2-morphisms $\triangleleft:f\circ p\to f$, $\rho:f\to f\circ p$, $\triangleright:q\circ f\to f$ and $\lambda:f\to q\circ f$ in $\mathbb{B}$ such that

$$(\mathrm{id}_f\circ m_p)\cdot(\rho\circ\mathrm{id}_p)=(\triangleleft\circ\mathrm{id}_p)\cdot(\mathrm{id}_f\circ\Delta_p)=\rho\cdot\triangleleft,\ \triangleleft\cdot\rho=\mathrm{id}_f,$$
$$(\mathrm{id}_q\circ\triangleright)\cdot(\Delta_q\circ\mathrm{id}_f)=(m_q\circ\mathrm{id}_f)\cdot(\mathrm{id}_q\circ\lambda)=\lambda\cdot\triangleright,\ \triangleright\cdot\lambda=\mathrm{id}_f,$$
$$\triangleright\cdot(\mathrm{id}_q\circ\triangleleft)=\triangleleft\cdot(\triangleright\circ\mathrm{id}_p)\text{ and }(\lambda\circ\mathrm{id}_p)\cdot\rho=(\mathrm{id}_q\circ\rho)\cdot\lambda.$$

Whenever it is clear from context, we will denote $(f,\triangleleft,\rho,\triangleright,\lambda)$ as $f:A_p\to B_q$ or even just as f. Graphically, this can be represented as follows.

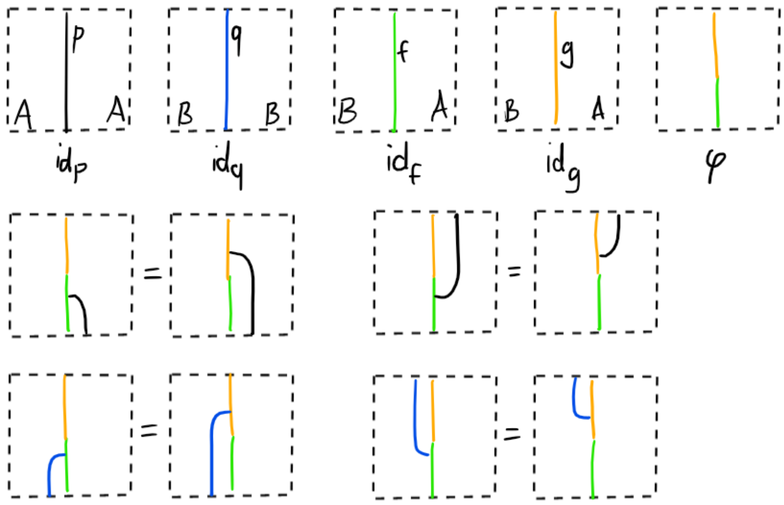


We can also think of a 1-morphism in $\widehat{\mathbb{B}}$ as a bimodule over special non-unital non-counital Frobenius algebras with $\triangleleft$ and ρ being the the right (co)action and $\triangleright$ and λ being the left (co)action.

- A 2-morphism between $(f,\triangleleft_f,\rho_f,\triangleright_f,\lambda_f)$ and $(g,\triangleleft_g,\rho_g,\triangleright_g,\lambda_g)$ is a 2-morphism $\varphi\colon f \to g$ in $\mathbb{B}$ such that

$$\varphi\cdot\triangleleft_f=\triangleleft_g\cdot(\varphi\circ \mathrm{id}_p),\ (\varphi\circ\mathrm{id}_p)\cdot\rho_f=\rho_g\cdot\varphi,$$
$$\varphi\cdot\triangleright_f=\triangleright_g\cdot(\mathrm{id}_q\circ\varphi)\text{ and }(\mathrm{id}_q\circ\varphi)\cdot\lambda_f=\lambda_g\cdot\varphi.$$

This can be graphically represented in the following way.

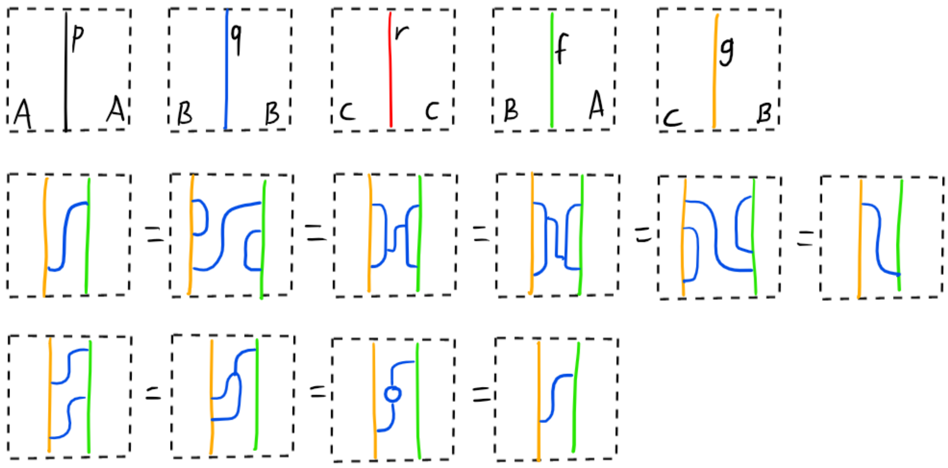


- Vertical composition, i.e., composition of 2-morphisms is simply given by the composition of 2-morphisms in $\mathbb{B}$. It follows that identity 2-morphisms are given by the identity 2-morphisms in $\mathbb{B}$. This turns $\widehat{\mathbb{B}}(A_p,B_q)$ into a category.

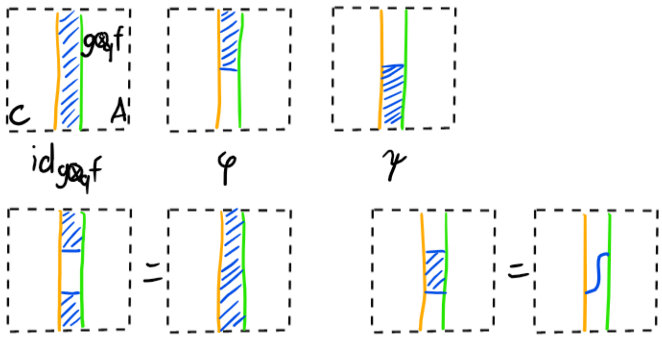
- Composition of 1-morphisms (or horizontal composition) is defined as follows. Let $f\colon A_p \to B_q$ and $g\colon B_q \to C_r$ be 1-morphisms in $\widehat{\mathbb{B}}$. We can now form the idempotent

$$\kappa_{g\circ f}=(\mathrm{id}_g\circ\triangleright_f)\cdot(\rho_g\circ\mathrm{id}_f)=(\triangleleft_g\circ\mathrm{id}_f)\cdot(\mathrm{id}_g\circ\lambda_f)\colon g\circ f\to g\circ f.$$

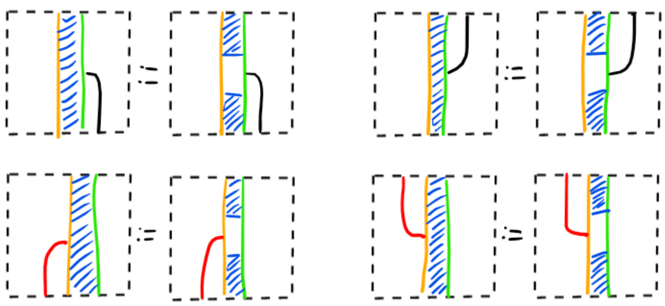
We define the composition of f and g as the splitting of this idempotent. We can graphically check that these two ways of writing this morphism do in fact coincide and that it forms an idempotent.



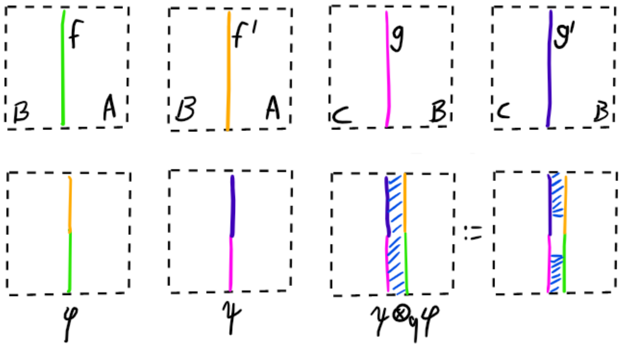
Since $\mathbb{B}$ is locally idempotent complete, we know that the idempotent $\kappa_{g\circ f}$ admits a splitting and thus we can choose a 1-morphism $g\otimes_q f : A \to C$ and 2-morphisms $\varphi : g\circ f \to g\otimes_q f$ and $\psi : g\otimes_q f \to g\circ f$ such that $\varphi\cdot\psi = \mathrm{id}_{g\otimes_q f}$ and $\psi\cdot\varphi = \kappa_{g\circ f}$, which we write graphically in the following way.



We can now define right-p (co)actions, and left-r (co)actions on $g\otimes_q f$ as follows.



This in fact turns it into a morphism $g\otimes_q f : A_p \to C_r$. Horizontal composition of 2-morphisms is defined in the following way.

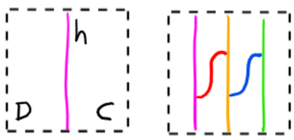


This turns composition into a functor

$$c_{A_p,B_q,C_r}:\widehat{\mathbb{B}}(B_q,C_r)\times\widehat{\mathbb{B}}(A_p,B_q)\to\widehat{\mathbb{B}}(A_p,C_r).$$

It is important to note here that when defining the Karoubi completion of a given bicategory $\mathbb{B}$, we have to make a choice for each pair of morphisms to be able to define this functor. All possible choices will lead to equivalent bicategories, but we have to make these choices none the less.

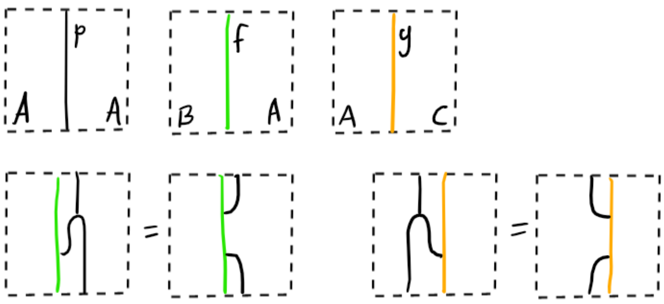
Let D_s be another object in $\widehat{\mathbb{B}}$ and $h:C_r\to D_s$ another 1-morphism. The 1-morphisms $(h\otimes_r g)\otimes_q f:A_p\to D_s$ and $h\otimes_r(g\otimes_q f):A_p\to D_s$ are now both splittings of the idempotent given by



and thus we have a unique isomorphism $a_{h,g,f}:(h\otimes_r g)\otimes_q f\to h\otimes_r(g\otimes_q f)$ and in general we get a natural isomorphism

$$a_{A_p,B_q,C_r,D_s}:c_{A_p,B_q,D_s}(c_{B_q,C_r,D_s}\times\mathrm{id}_{\widehat{\mathbb{B}}(A_p,B_q)})\to c_{A_p,C_r,D_s}(\mathrm{id}_{\widehat{\mathbb{B}}(C_r,D_s)}\times c_{A_p,B_q,C_r}).$$

- The identity 1-morphism on A_p is given by $(p,m_p,\Delta_p,m_p,\Delta_p)$. Let $f:A_p\to B_q$ and $g:C_r\to A_p$ be 1-morphism. We now have the following.



Since splittings are unique up to unique isomorphism, we have isomorphisms $l_g : p \otimes_p g \to g$ and $r_f : f \otimes_p p \to f$. These form natural isomorphism

$$l_{C_r,A_p} : c_{C_r,A_p,A_p}(p,-) \to \mathrm{id}_{\widehat{\mathbb{B}}(C_r,A_p)} \text{ and}$$
$$r_{A_p,b_q} : c_{A_p,A_p,B_q}(-,p) \to \mathrm{id}_{\widehat{\mathbb{B}}(A_p,B_q)}.$$

- Since both the associator and left and right unitors are given by unique isomorphisms between colimits, it follows immediately that the triangle and pentagon axiom both have to hold.

We first will show that $\widehat{\mathbb{B}}$ is a completion of $\mathbb{B}$ in the sense that $\mathbb{B}$ embeds into $\widehat{\mathbb{B}}$ and is equivalent to it if $\mathbb{B}$ was already idempotent complete.

Proposition 3.7. *For every locally idempotent complete bicategory $\mathbb{B}$, there exists a fully faithful pseudofunctor $\iota_{\mathbb{B}} : \mathbb{B} \to \widehat{\mathbb{B}}$. If $\mathbb{B}$ is furthermore idempotent complete, this functor is an equivalence.*

Proof. We define $\iota_{\mathbb{B}}$ to map the object A in $\mathbb{B}$ onto $(A, \mathrm{id}_A, \mathrm{id}_{\mathrm{id}_A}, \mathrm{id}_{\mathrm{id}_A})$ in $\widehat{\mathbb{B}}$. $\iota_{\mathbb{B}}$ maps 1-morphisms $f : A \to B$ in $\mathbb{B}$ onto the the morphism $(f, \mathrm{id}_f, \mathrm{id}_f, \mathrm{id}_f, \mathrm{id}_f) : A_{\mathrm{id}_A} \to B_{\mathrm{id}_B}$ and 2-morphisms $\varphi : f \to g$ in $\mathbb{B}$ onto $\varphi : f \to g$ in $\widehat{\mathbb{B}}$.

We now note that, for objects A, B in $\mathbb{B}$, every morphism in $\widehat{\mathbb{B}}$ between A_{id_A} and B_{id_B} is of the form $(f, \mathrm{id}_f, \mathrm{id}_f, \mathrm{id}_f, \mathrm{id}_f)$ where f is an arbitrary morphism $f : A \to B$ in $\mathbb{B}$. Since a 2-morphism between 1-morphisms of the form $(f, \mathrm{id}_f, \mathrm{id}_f, \mathrm{id}_f, \mathrm{id}_f)$ and $(g, \mathrm{id}_g, \mathrm{id}_g, \mathrm{id}_g, \mathrm{id}_g)$ is also just an arbitrary 2-morphism $\varphi : f \to g$, we see that $\iota_{\mathbb{B}}$ is fully faithful.

Now assume that $\mathbb{B}$ is idempotent complete and let (A, p, m, Δ) be an object in $\widehat{\mathbb{B}}$. We want to show that $\iota_{\mathbb{B}}$ is essentially surjective, i.e., there exists some B in $\mathbb{B}$ such that $A_p \simeq \iota_{\mathbb{B}} B$. We know that the 2-idempotent (A, p, m, Δ) splits in $\mathbb{B}$, which means there exists an object B in $\mathbb{B}$, 1-morphisms $f : A \to B$ and $g : B \to A$ in $\mathbb{B}$, 2-morphisms $\varphi : f \circ g \to \mathrm{id}_B$, $\psi : \mathrm{id}_B \to f \circ g$ and an isomorphism $\gamma : g \circ f \to p$ such that $\varphi \cdot \psi = \mathrm{id}_{\mathrm{id}_B}$, $m = \gamma \cdot (\mathrm{id}_g \circ \varphi \circ \mathrm{id}_f) \cdot (\gamma^{-1} \circ \gamma^{-1})$ and $\Delta = (\gamma \circ \gamma) \cdot (\mathrm{id}_g \circ \psi \circ \mathrm{id}_f) \cdot \gamma^{-1}$.

We can now define the morphisms

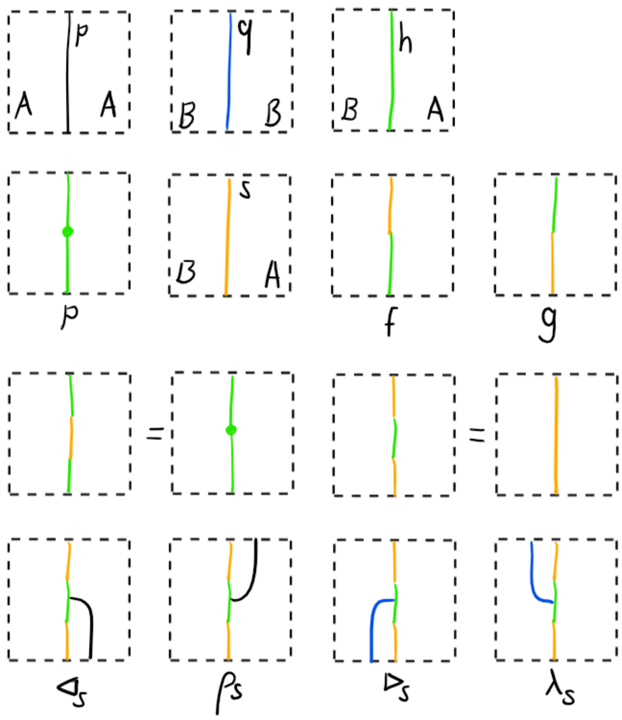
$$(f, (\varphi \circ \mathrm{id}_f) \cdot (\mathrm{id}_f \circ \gamma^{-1}), (\mathrm{id}_f \circ \gamma) \cdot (\psi \circ \mathrm{id}_f), \mathrm{id}_f, \mathrm{id}_f) : A_p \to B_{\mathrm{id}_B} \text{ and}$$
$$(g, \mathrm{id}_g, \mathrm{id}_g, (\mathrm{id}_g \circ \varphi) \cdot (\gamma^{-1} \circ \mathrm{id}_g), (\gamma \circ \mathrm{id}_g) \cdot (\mathrm{id}_g \circ \psi)) : B_{\mathrm{id}_B} \to A_p$$

and, with these, we now have $g \otimes_{\mathrm{id}_B} f \cong g \circ f \cong p = \mathrm{id}_{A_p}$ and $f \otimes_p g \cong \mathrm{id}_B$. Thus it follows that $A_p \simeq B_{\mathrm{id}_B}$ and $\iota_{\mathbb{B}} : \mathbb{B} \to \widehat{\mathbb{B}}$ is an equivalence of bicategories. □

For $\widehat{\mathbb{B}}$ to be the completion of $\mathbb{B}$ under splitting 2-idempotents, we want $\widehat{\mathbb{B}}$ itself to be idempotent complete, which we will show in the following.

Proposition 3.8. *For every locally idempotent complete bicategory $\mathbb{B}$, the bicategory $\widehat{\mathbb{B}}$ is idempotent complete.*

Proof. We first need to show that $\widehat{\mathbb{B}}$ is locally idempotent complete. Let A_p and B_q be objects in $\widehat{\mathbb{B}}$, $(h, \triangleleft_h, \rho_h, \triangleright_h, \lambda_h) : A_p \to B_q$ a 1-morphism and $p : h \to h$ an idempotent 2-morphism. Since $\mathbb{B}$ is locally idempotent complete, we know that the idempotent p splits into 2-morphism $f : h \to s$ and $g : s \to h$ such that $gf = p$ and $fg = id_s$ where $s : A \to B$ is a 1-morphism in $\mathbb{B}$. Note that, a prior, s is not a morphism in $\widehat{\mathbb{B}}$ but we can turn s into a morphism $s : A_p \to B_q$ with the following left and right (co)action.



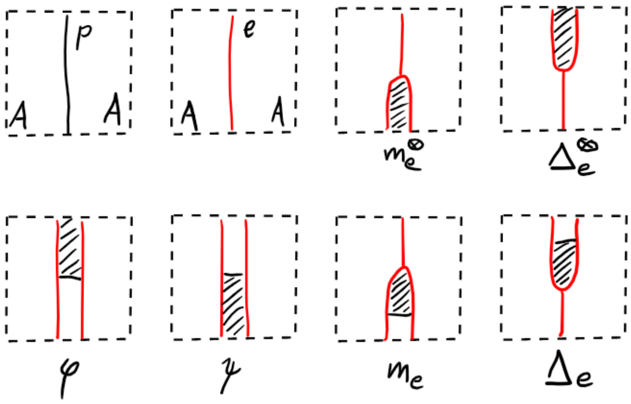
With this definition of s, f and g also become 2-morphisms in $\widehat{\mathbb{B}}$ and therefore split the idempotent p. Thus $\widehat{\mathbb{B}}$ is locally idempotent complete.

Now, let (A, p, m, Δ) be an object in $\widehat{\mathbb{B}}$ and $(e, \triangleleft, \rho, \triangleright, \lambda)$ a 2-idempotent on A_p, which means we have morphisms $m_e^{\otimes} : e \otimes_p e \to e$ and $\Delta_e^{\otimes} : e \to e \otimes_p e$ such that

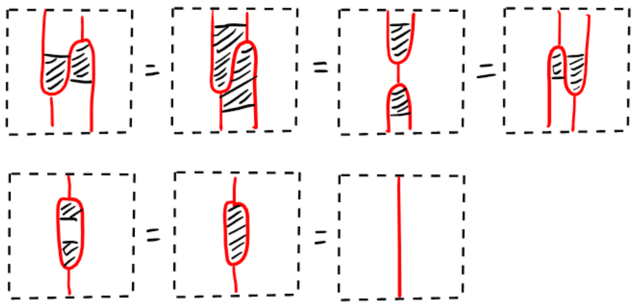
$$(id_e \otimes_p m_e^{\otimes}) \cdot (\Delta_e^{\otimes} \otimes_p id_e) = (m_e^{\otimes} \otimes_p id_e) \cdot (id_e \otimes_p \Delta_e^{\otimes}) = \Delta_e^{\otimes} \cdot m_e^{\otimes} \text{ and}$$
$$m_e^{\otimes} \cdot \Delta_e^{\otimes} = id_e.$$

We want to show that this 2-idempotent splits. From the definition of $e \otimes_p e$ we have 2-morphisms $\varphi : e \circ e \to e \otimes_p e$ and $\psi : e \otimes_p e \to e \circ e$ such that $\varphi \cdot \psi = id_{e \otimes_p e}$ and $\psi \cdot \varphi = \kappa_{e \circ e}$.

Combining these we get morphisms $m_e = m_e^{\otimes} \cdot \varphi : e \circ e \to e$ and $\Delta_e = \psi \cdot \Delta_e^{\otimes} : e \to e \circ e$. Graphically, we can represent these in the following way.



This defines a 2-idempotent in $\mathbb{B}$ as we will show graphically.



Therefore (A, e, m_e, Δ_e) is an object in $\widehat{\mathbb{B}}$ and furthermore we can now define morphisms $(e, \triangleleft, \rho, m_e, \Delta_e) : A_p \to A_e$ and $(e, m_e \Delta_e, \triangleright, \lambda) : A_e \to A_p$. Composing them in $\widehat{\mathbb{B}}$, we get $e \otimes_e e \cong e : A_p \to A_p$ and $e \otimes_p e : A_e \to A_e$. As we have morphisms $m_e^{\otimes}$: $e \otimes_p e \to e = \mathrm{id}_{A_e}$ and $\Delta_e^{\otimes} : \mathrm{id}_{A_e} = e \to e \otimes_p e$, we have shown that $(e, \triangleleft, \rho, \triangleright, \lambda)$ splits in $\widehat{\mathbb{B}}$ and thus $\widehat{\mathbb{B}}$ is idempotent complete. $\square$

Since a 2-idempotent $p : A \to A$ in $\mathbb{B}$ defines a 2-idempotent $p : A_{\mathrm{id}_A} \to A_{\mathrm{id}_A}$ in $\widehat{\mathbb{B}}$, we also get the following statement.

Remark 3.9. For an object A_p in $\widehat{\mathbb{B}}$, A_p is a splitting of the 2-idempotent p on A_{id_A}.

Lastly, we want to show that the Karoubi completion $\widehat{\mathbb{B}}$ is universal among all possible idempotent completions of $\mathbb{B}$, which is why we are able to call it *the* idempotent completion of $\mathbb{B}$. What we mean by universal is that for each pseudofunctor F from $\mathbb{B}$ into an arbitrary idempotent complete bicategory $\mathbb{C}$, there is a pseudofunctor $F' : \widehat{\mathbb{B}} \to \mathbb{C}$ such that $F' \iota_{\mathbb{B}} \cong F$.

Theorem 3.10. *For each locally idempotent complete bicategory $\mathbb{B}$ and idempotent complete bicategory $\mathbb{C}$, we have an equivalence of bicategories*

$$\mathrm{Bicat}(\widehat{\mathbb{B}},\mathbb{C}) \simeq \mathrm{Bicat}(\mathbb{B},\mathbb{C})$$

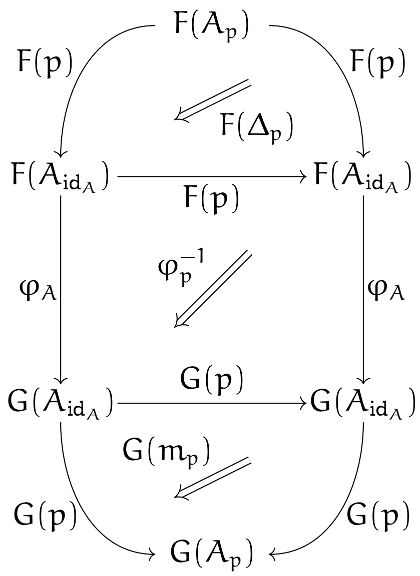
induced by precomposing with $\iota_{\mathbb{B}}$.

Proof. First, we want to show that for every pseudofunctor $F:\mathbb{B}\to\mathbb{C}$, there exists a pseudofunctor $F':\widehat{\mathbb{B}}\to\mathbb{C}$ such that $F'\iota_{\mathbb{B}}\simeq F$, i.e., precomposition with $\iota_{\mathbb{B}}$ is essentially surjective. For a given pseudofunctor $F:\mathbb{B}\to\mathbb{C}$ we can define the pseudofunctor $\widehat{F}:\widehat{\mathbb{B}}\to\widehat{\mathbb{C}}$ which maps a 2-idempotent (A,p,m,Δ) onto the 2-idempotent $(F(A),F(p),F(m),F(\Delta))$ and acts analogously on 1-morphisms and 2-morphisms. The following diagram now commutes up to equivalence

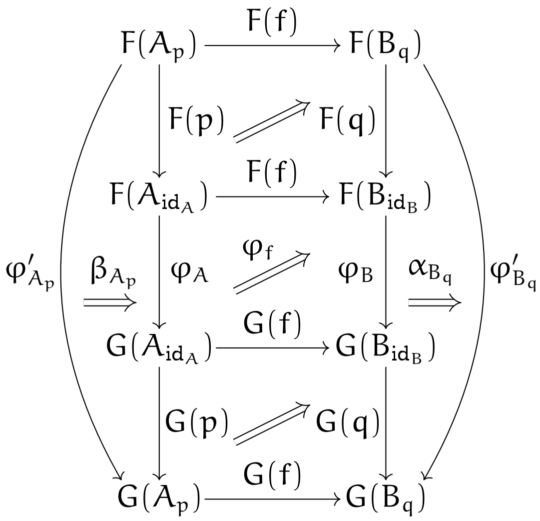
$$\begin{array}{ccc} \mathbb{B} & \xrightarrow{F} & \mathbb{C} \\ \downarrow{\scriptstyle \iota_{\mathbb{B}}} & & \downarrow{\scriptstyle \iota_{\mathbb{C}}} \\ \widehat{\mathbb{B}} & \xrightarrow{\widehat{F}} & \widehat{\mathbb{C}} \end{array}$$

Since $\mathbb{C}$ is an idempotent complete bicategory, $\iota_{\mathbb{C}}$ is an equivalence and we can choose an inverse $\iota_{\mathbb{C}}^{-1}$. We now define $F'=\iota_{\mathbb{C}}^{-1}\widehat{F}$ and have $F'\iota_{\mathbb{B}}=\iota_{\mathbb{C}}^{-1}\widehat{F}\iota_{\mathbb{B}}\simeq\iota_{\mathbb{C}}^{-1}\iota_{\mathbb{C}}F\simeq F$.

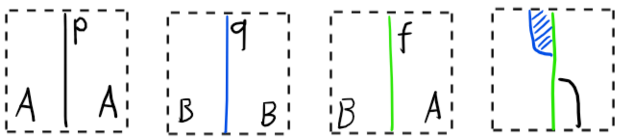
Secondly, we need to show that precomposition with $\iota_{\mathbb{B}}$ is essentially surjective on Hom-categories. Let $F,G:\widehat{\mathbb{B}}\to\mathbb{C}$ be pseudofunctors. We want to show that for each strong transformation $\varphi:F\iota_{\mathbb{B}}\to G\iota_{\mathbb{B}}$, we can find a strong transformation $\varphi':F\to G$ such that $\varphi'\iota_{\mathbb{B}}\cong\varphi$. For a given $\varphi:F\iota_{\mathbb{B}}\to G\iota_{\mathbb{B}}$, we define such a φ' by defining an idempotent and then setting φ'_{A_p} to be its splitting. This idempotent $\gamma_{A_p}:G(p)\varphi_A F(p)\to G(p)\varphi_A F(p)$ is given via



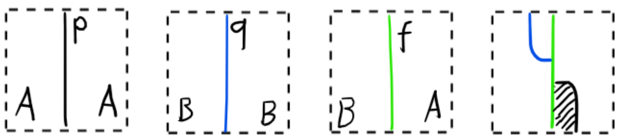
and we now have a splitting $\varphi'_{A_p}: F(A_p) \to G(A_p)$ along with morphisms $\alpha_{A_p}: G(p)\varphi_A F(p) \to \varphi'_{A_p}$ and $\beta_{A_p}: \varphi'_{A_p} \to G(p)\varphi_A F(p)$ which satisfy $\alpha_{A_p}\beta_{A_p} = \mathrm{id}_{\varphi'_{A_p}}$ and $\beta_{A_p}\alpha_{A_p} = \gamma_{A_p}$. We define φ'_f on a morphism $f: A_p \to B_q$ via



where the top morphism is given by the image of



under F and the bottom morphism is given by the image of

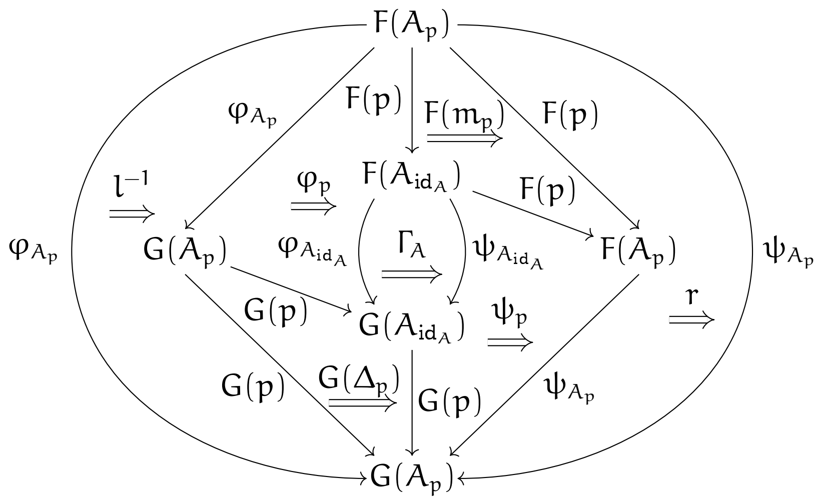


under G. This turns $\varphi': G \to F$ into a strong transformation. We now need to show that $\varphi'\iota_{\mathbb{B}} \cong \varphi$, i.e. $\varphi'_{A_{\mathrm{id}_A}} \cong \varphi_A$. If we look at the idempotent $\gamma_{A_{\mathrm{id}_A}}$ we see that it splits via φ_A and thus $\varphi'_{A_{\mathrm{id}_A}} \cong \varphi_A$.

Last, we need to show that precomposition with $\iota_{\mathbb{B}}$ is fully faithful on Hom-categories. Let $F, G: \widehat{\mathbb{B}} \to \mathbb{C}$ be pseudofunctors and $\varphi, \psi: F \to G$ strong transformations. We want to show that we have an isomorphism

$$\mathbf{Bicat}(\widehat{\mathbb{B}}, \mathbb{C})(F, G)(\varphi, \psi) \cong \mathbf{Bicat}(\mathbb{B}, \mathbb{C})(F\iota_{\mathbb{B}}, G\iota_{\mathbb{B}})(\varphi\iota_{\mathbb{B}}, \psi\iota_{\mathbb{B}}).$$

Let $\Gamma: \varphi\iota_{\mathbb{B}} \to \psi\iota_{\mathbb{B}}$ be a modification. We can construct a modification $\Gamma': \varphi \to \psi$ with components $\Gamma'_{A_p}: \varphi_{A_p} \to \psi_{A_p}$ via the following.



This defines a modification $\Gamma' \colon F \to G$ and we can see that this construction defines an inverse on the level of modifications. Thus precomposition with $\iota_{\mathbb{B}}$ is fully faithful on Hom-categories and we have an equivalence

$$\mathbf{Bicat}(\widehat{\mathbb{B}}, \mathbb{C}) \simeq \mathbf{Bicat}(\mathbb{B}, \mathbb{C}).$$

□

Remark 3.11. Without going into technical details, theorem 3.10 is a consequence of the fact that idempotent completion should form a left adjoint trifunctor $\widehat{(-)} \colon \mathbf{Bicat}_{\mathrm{lic}} \to \mathbf{Bicat}_{\mathrm{ic}}$ from the tricategory of locally idempotent complete bicategories, pseudofunctors, strong transformations and modifications into the full subtricategory of idempotent complete bicategories, analogously to theorem 1.12.

4 Cauchy Completion

We are now going to define the Cauchy completion of a bicategory, which is its completion under absolute weighted colimits. We will see, that it is universal among all bicategories that have this property. Before we get to this, we will show that a 2-idempotent splitting in a locally idempotent bicategory is a weighted colimit. Since 2-idempotent splittings are preserved by every pseudofunctor, they in fact define absolute weighted colimits.

4.1 2-Idempotent Splittings as Absolute Weighted Colimits

We briefly recall definition 3.4 of the free walking 2-idempotent splitting $\spadesuit_2$. $\spadesuit_2$ has two objects X and Y and a 2-idempotent on X which splits via Y given by morphisms $f\colon X\to Y$, $g\colon Y\to X$, $\varphi\colon fg\to \mathrm{id}_Y$ and $\psi\colon \mathrm{id}_Y\to fg$. The free walking 2-idempotent $\clubsuit_2$ is the full subbicategory of $\spadesuit_2$ on X.

Proposition 4.1. *Let $\mathbb{B}$ be a locally idempotent complete bicategory and let (A,p,m,Δ) be a 2-idempotent which corresponds to the pseudofunctor $F\colon\clubsuit_2\to\mathbb{B}$. The weighted colimit of F along the weight $\spadesuit_2(\iota-,Y)\colon\clubsuit_2\to \mathrm{Cat}_{\mathrm{ic}}$ determines a splitting of A_p and every splitting determines a colimit of F weighted by $\spadesuit_2(\iota-,Y)$.*

Proof. Let B in $\mathbb{B}$ be a weighted colimit with colimiting strong transformation $\lambda\colon\spadesuit_2(\iota-,Y)\to\mathbb{B}(F-,B)$. We want to show that this defines a splitting of the idempotent determined by F. We have a functor $\lambda_X\colon\spadesuit_2(X,Y)\to\mathbb{B}(A,B)$ and a 1-morphism $\tilde{f}=\lambda_X(f)\colon A\to B$.

We can define a strong transformation $\kappa\colon\spadesuit_2(\iota-,Y)\to\mathbb{B}(F-,A)$ via

$$\begin{gathered}\kappa_X\colon\spadesuit_2(X,Y)\to\mathbb{B}(A,A)\\ f\mapsto p\\ (\varphi\mathrm{id}_f\colon fgf\to f)\mapsto(m\colon p^2\to p)\\ (\psi\mathrm{id}_f\colon f\to fgf)\mapsto(\Delta\colon p\to p^2)\end{gathered}$$

This fully defines κ up to isomorphism since the action of $\clubsuit_2(X,X)$ on $\spadesuit_2(X,Y)$ generates every other object and morphism in $\spadesuit_2(X,Y)$. By the universal property of the weighted colimit, we now have a morphism $\tilde{g}\colon B\to A$ such that $\tilde{g}_*\lambda\cong\kappa$ given by a modification $\Gamma\colon\tilde{g}_*\lambda\to\kappa$. This modification also gives us the desired isomorphism $\gamma=\Gamma_{X,f}\colon\tilde{g}\tilde{f}\to p$.

We now also need to find morphisms $\tilde{\varphi}\colon\tilde{f}\tilde{g}\to\mathrm{id}_B$ and $\tilde{\psi}\colon\mathrm{id}_B\to\tilde{f}\tilde{g}$ such that $\tilde{\varphi}\tilde{\psi}=$

$\mathrm{id}_{\mathrm{id}_B}$. For this we use the equivalence

$$\lambda_* : \mathbb{B}(B,B) \to \mathbf{Bicat}(\clubsuit_2, \mathbf{Cat}_{\mathrm{ic}})(\spadesuit_2(\iota -, Y), \mathbb{B}(F-,B))$$

by which we know it is sufficient to find modifications $\Phi : (\tilde f\tilde g)_*\lambda \to \lambda$ and $\Psi : \lambda \to (\tilde f\tilde g)_*\lambda$ such that $\Phi\Psi = \mathrm{id}_\lambda$. Similar to above, it is sufficient to give $\Phi_{X,f}$ and $\Psi_{X,f}$ to define Φ and Ψ up to isomorphism. These we can define by

$$\tilde f\tilde g\lambda_X(f) = \tilde f\tilde g\tilde f \xrightarrow[\tilde f\gamma]{\cong} \tilde f p \xrightarrow[\lambda_{gf}]{\cong} \lambda_X(fgf) \underset{\lambda_X(\psi f)}{\overset{\lambda_X(\varphi f)}{\rightleftarrows}} \lambda_X(f) = \tilde f$$

where λ_{gf} is given by

$$\begin{array}{ccc} \spadesuit_2(X,Y) & \xrightarrow{(gf)^*} & \spadesuit_2(X,Y) \\ {\scriptstyle\lambda_X}\downarrow & \overset{\lambda_{gf}}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle\lambda_X} \\ \mathbb{B}(A,B) & \xrightarrow[p^*]{} & \mathbb{B}(A,B) \end{array}$$

One can clearly see that these satisfy $\Phi\Psi = \mathrm{id}_\lambda$ and thus they induce morphisms $\tilde\varphi : \tilde f\tilde g \to \mathrm{id}_B$ and $\tilde\psi : \mathrm{id}_B \to \tilde f\tilde g$ such that $\tilde\varphi\tilde\psi = \mathrm{id}_{\mathrm{id}_B}$. We now need to check the compatibility between $\tilde\varphi$ and $\tilde\psi$ and m and Δ. This we can represent via the diagram

$$\begin{array}{ccc} \tilde g\tilde f\tilde g\tilde f & \underset{\tilde g\tilde\psi\tilde f}{\overset{\tilde g\tilde\varphi\tilde f}{\rightleftarrows}} & \tilde g\tilde f \\ \downarrow{\scriptstyle\gamma\gamma} & & \downarrow{\scriptstyle\gamma} \\ p^2 & \underset{\Delta}{\overset{m}{\rightleftarrows}} & p \end{array}$$

We know that the diagram

$$\begin{array}{ccccccc} \tilde g\lambda_X(f)\tilde g\tilde f & \xrightarrow{\tilde g\lambda_X(f)\gamma} & \tilde g\lambda_X(f)p & \xrightarrow{\tilde g\lambda_{gf}} & \tilde g\lambda_X(fgf) & \underset{\tilde g\lambda_X(\psi f)}{\overset{\tilde g\lambda_X(\varphi f)}{\rightleftarrows}} & \tilde g\lambda_X(f) \\ {\scriptstyle\Gamma_{X,f}\widetilde{gf}}\downarrow & & {\scriptstyle\Gamma_{X,f}p}\downarrow & & {\scriptstyle\Gamma_{X,fgf}}\downarrow & & \downarrow{\scriptstyle\Gamma_{X,f}} \\ \kappa_X(f)\tilde g\tilde f & \xrightarrow{\kappa_X(f)\gamma} & \kappa_X(f)p & \xrightarrow{\kappa_{gf}} & \kappa_X(fgf) & \underset{\kappa_X(\psi f)}{\overset{\kappa_X(\varphi f)}{\rightleftarrows}} & \kappa_X(f) \end{array}$$

commutes and this is the same diagram as the following

$$\begin{tikzcd}
\tilde{g}\tilde{f}\tilde{g}\tilde{f} \arrow[r, "\tilde{g}\tilde{f}\gamma"] \arrow[d, "\gamma\tilde{g}\tilde{f}"'] & \tilde{g}\tilde{f}p \arrow[r, "\tilde{g}\lambda_{gf}"] \arrow[d, "\gamma p"'] & \tilde{g}\lambda_X(fgf) \arrow[r, bend left, "\tilde{g}\lambda_X(\varphi f)"] \arrow[d, "\Gamma_{X,fgf}"'] & \tilde{g}\tilde{f} \arrow[l, bend left, "\tilde{g}\lambda_X(\psi f)"] \arrow[d, "\gamma"] \\
p\tilde{g}\tilde{f} \arrow[r, "p\gamma"] & p^2 \arrow[r, "="] & p^2 \arrow[r, bend left, "m"] & p \arrow[l, bend left, "\Delta"]
\end{tikzcd}$$

which gives us the desired result.

Now let $(A,B,f,g,\varphi,\psi,\gamma)$ be a splitting of A_p. First, we can define a strong transformation

$$\lambda\colon\spadesuit_2(\iota-,Y)\to\mathbb{B}(F-,B)$$

by $\lambda_X(f)=f\colon A\to B$. We now need to show that it is colimiting, i.e.,

$$\lambda^*\colon\mathbb{B}(B,C)\to\mathbf{Bicat}(\clubsuit_2,\mathbf{Cat}_{ic})(\spadesuit_2(\iota-,Y),\mathbb{B}(F-,C))$$

defines an equivalence of categories for each object C in $\mathbb{B}$. Let $\kappa\colon\spadesuit_2(\iota-,Y)\to\mathbb{B}(F-,C)$ be another strong transformation which is determined by $\kappa_X(f)\colon A\to C$. We can now form the morphism $\kappa_X(f)\otimes_p g\colon B\to C$ with which we can form the strong transformation $(\kappa_X(f)\otimes_p g)_*\lambda$. We now have

$$\kappa_X(f)\otimes_p g\lambda_X(f)=\kappa_X(f)\otimes_p gf\cong\kappa_X(f)\otimes_p p\cong\kappa_X(f)$$

which implies $(\kappa_X(f)\otimes_p g)_*\lambda\cong\kappa$ and thus λ^* is essentially surjective. Now let $h,h'\colon B\to C$ be 1-morphisms in $\mathbb{B}$. We now want to show that λ^* induces an ismorphism

$$\mathbb{B}(B,C)(h,h')\cong\mathbf{Bicat}(\clubsuit_2,\mathbf{Cat}_{ic})(\spadesuit_2(\iota-,Y),\mathbb{B}(F-,C))(h_*\lambda,h'_*\lambda).$$

Let $\Gamma\colon h_*\lambda\to h'_*\lambda$ be a strong transformation. We can define a morphism $\theta\colon h\to h'$ via

$$\begin{tikzcd}
h \arrow[r, "\theta"] \arrow[d, "\cong"] & h' \arrow[d, "\cong"] \\
hf\otimes_p g \arrow[r, "\Gamma_{X,f}\otimes_p g"] & h'f\otimes_p g
\end{tikzcd}$$

and we have $\theta_*\lambda=\Gamma$ since

$$\begin{tikzcd}
hf \arrow[r, "\Gamma_{X,f}=\theta f"] \arrow[d, "\cong"] & h'f \arrow[d, "\cong"] \\
hf\otimes_p gf \arrow[r, "\Gamma_{X,f}\otimes_p gf"] & h'f\otimes_p gf
\end{tikzcd}$$

and thus λ^* is full. Lastly we want to show that it is faithful. Let $\theta,\theta'\colon h\to h'$ be 2-morphisms such that $\theta_*\lambda=\theta'_*\lambda$. We now have

$$\begin{array}{ccc} h & \overset{\theta}{\underset{\theta'}{\rightrightarrows}} & h' \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\ hf\otimes_p g & \overset{\theta f\otimes_p g}{\underset{\theta' f\otimes_p g}{\longrightarrow}} & h'f\otimes_p g \end{array}$$

and thus λ^* is faithful. All put together, we have shown that λ^* defines an equivalence of categories and therefore defines a weighted colmit of F along $\spadesuit_2(\iota-,Y)$. $\square$

Corollary 4.2. *Any two splittings of the same 2-idempotent in a locally idempotent complete bicategory are equivalent.*

Corollary 4.3. $\spadesuit_2(\iota-,Y):\clubsuit_2\to\mathrm{Cat}_{\mathrm{ic}}$ *is an absolute weight.*

Corollary 4.4. *Let* $\mathbb{B}$ *be a bicategory. The bicategories* $\mathrm{Cat}_{\mathrm{ic}}$, $\mathrm{Cat}_{\mathrm{ic}}^{\mathrm{op}}$ *and* $\mathrm{Psh}_{\mathrm{ic}}(\mathbb{B})$ *are 2-idempotent complete.*

Proof. We have already seen that $\mathbf{Cat}_{\mathrm{ic}}$, $\mathbf{Cat}_{\mathrm{ic}}^{\mathrm{op}}$ and $\mathbf{Psh}_{\mathrm{ic}}(\mathbb{B})$ are all locally idempotent complete. Since $\mathbf{Cat}_{\mathrm{ic}}$ and $\mathbf{Psh}_{\mathrm{ic}}(\mathbb{B})$ are cocomplete and splittings of a 2-idempotents are given by a weighted colimit, they are also 2-idempotent complete.

One can show, that the data of a 2-idempotent in $\mathbf{Cat}_{\mathrm{ic}}^{\mathrm{op}}$ is the same data as that of a 2-idempotent in $\mathbf{Cat}_{\mathrm{ic}}$ and furthermore a splitting of this idempotent in $\mathbf{Cat}_{\mathrm{ic}}^{\mathrm{op}}$ defines a splitting of it in $\mathbf{Cat}_{\mathrm{ic}}$ and vice versa. This implies that $\mathbf{Cat}_{\mathrm{ic}}^{\mathrm{op}}$ is also 2-idempotent complete. $\square$

4.2 Cauchy Completion of a Bicategory

Finally, we can define the bicategorical analogue of Cauchy completion.

Definition 4.5. (Cauchy completion) We call an object A in a locally idempotent complete bicategory $\mathbb{B}$ *tiny* if the pseudofunctor $\mathbb{B}(A,-):\mathbb{B}\to\mathbf{Cat}_{\mathrm{ic}}$ is cocontinuous, i.e., preserves all weighted colimits.

Let $\mathbb{B}$ be a locally idempotent complete bicategory. The *Cauchy completion* of $\mathbb{B}$ is defined to be the full subbicategory of tiny objects in $\mathbf{Psh}_{\mathrm{ic}}(\mathbb{B})$, i.e., an object in this bicategory is a pseudofunctor $S:\mathbb{B}\to\mathbf{Cat}_{\mathrm{ic}}^{\mathrm{op}}$ such that $\mathbf{Psh}_{\mathrm{ic}}(\mathbb{B})(S,-):\mathbf{Psh}_{\mathrm{ic}}(\mathbb{B})\to\mathbf{Cat}_{\mathrm{ic}}$ is cocontinuous. We will denote this bicategory as $\mathbf{Psh}_{\mathrm{ic}}^{\mathrm{tn}}(\mathbb{B})$.

We first will show that $\mathbf{Psh}_{\mathrm{ic}}^{\mathrm{tn}}(\mathbb{B})$ is a completion of $\mathbb{B}$ in the sense that $\mathbb{B}$ embeds into $\mathbf{Psh}_{\mathrm{ic}}^{\mathrm{tn}}(\mathbb{B})$ and is equivalent to it if $\mathbb{B}$ was already complete under absolute weighted colimits. To do this we will prove the following propositions.

Proposition 4.6. *Representable presheaves are tiny.*

Proof. Let A be an object in $\mathbb{B}$, we will now show that $\mathbb{B}(-,A)$ is tiny. Let $F\colon \mathbb{J}\to\mathbf{Psh}_{\mathsf{ic}}(\mathbb{B})$ be a pseudofunctor and $W\colon\mathbb{J}\to\mathbf{Cat}_{\mathsf{ic}}^{\mathsf{op}}$ a weight with colimit $\mathrm{colim}_W F$, i.e., we have a strong transformation equivalence

$$\lambda^*\colon\mathbf{Psh}_{\mathsf{ic}}(\mathbb{B})(\mathrm{colim}_W F,-)\to\mathbf{Bicat}(\mathbb{J},\mathbf{Cat}_{\mathsf{ic}}^{\mathsf{op}})(W,\mathbf{Psh}_{\mathsf{ic}}(\mathbb{B})(F,-))$$

given by precomposing with a colimiting strong transformation

$$\lambda\colon W\to\mathbf{Psh}_{\mathsf{ic}}(\mathbb{B})(F-,\mathrm{colim}_W F).$$

We will now see that $\mathbf{Psh}_{\mathsf{ic}}(\mathbb{B})(\mathbb{B}(-,A),-)$ preserves this colimit. Applying this functor to λ yields a strong transformation

$$\begin{aligned}&\mathbf{Psh}_{\mathsf{ic}}(\mathbb{B})(\mathbb{B}(-,A),\lambda)\colon W\\ &\qquad\to\mathbf{Cat}_{\mathsf{ic}}(\mathbf{Psh}_{\mathsf{ic}}(\mathbb{B})(\mathbb{B}(-,A),F-),\mathbf{Psh}_{\mathsf{ic}}(\mathbb{B})(\mathbb{B}(-,A),\mathrm{colim}_W F)).\end{aligned}$$

We will now check that

$$\begin{aligned}&\mathbf{Psh}_{\mathsf{ic}}(\mathbb{B})(\mathbb{B}(-,A),\lambda)^*\colon\mathbf{Cat}_{\mathsf{ic}}(\mathbf{Psh}_{\mathsf{ic}}(\mathbb{B})(\mathbb{B}(-,A),\mathrm{colim}_W F),-)\\ &\qquad\to\mathbf{Bicat}(\mathbb{J},\mathbf{Cat}_{\mathsf{ic}}^{\mathsf{op}})(W,\mathbf{Cat}_{\mathsf{ic}}(\mathbf{Psh}_{\mathsf{ic}}(\mathbb{B})(\mathbb{B}(-,A),F-),-))\end{aligned}$$

defines an equivalence of pseudofunctors. Via the Yoneda lemma, the strong transformation $\mathbf{Psh}_{\mathsf{ic}}(\mathbb{B})(\mathbb{B}(-,A),\lambda)$ corresponds to the strong transformation $\lambda_A\colon W\to\mathbf{Cat}_{\mathsf{ic}}(F(-)(A),\mathrm{colim}_W F(A))$. Since weighted colimits in $\mathbf{Psh}_{\mathsf{ic}}(\mathbb{B})$ are computed point-wise, we know that λ_A has to define a weighted colimit which implies that $\mathbf{Psh}_{\mathsf{ic}}(\mathbb{B})(\mathbb{B}(-,A),\lambda)$ already had to have been colimiting. Thus the pseudofunctor $\mathbf{Psh}_{\mathsf{ic}}(\mathbb{B})(\mathbb{B}(-,A),-)$ preserves colimits and $\mathbb{B}(A,-)$ is tiny. □

Proposition 4.7. *Let $\mathbb{B}$ be a locally idempotent complete bicategory. Every tiny presheaf $S\colon\mathbb{B}\to\mathrm{Cat}_{\mathsf{ic}}^{\mathsf{op}}$ is a retract of a representable presheaf.*

Proof. We can express S as the weighted colimit of $よ_{\mathbb{B}}\colon\mathbb{B}\to\mathbf{Psh}_{\mathsf{ic}}(\mathbb{B})$ along the weight $S\colon\mathbb{B}\to\mathbf{Cat}_{\mathsf{ic}}^{\mathsf{op}}$ with a strong transformation $\lambda\colon S\to\mathbf{Psh}_{\mathsf{ic}}(\mathbb{B})(よ_{\mathbb{B}}-,S)$ given by the Yoneda lemma. Since S is tiny we can now apply the functor $\mathbf{Psh}_{\mathsf{ic}}(\mathbb{B})(S,-)$ to this colimit and we have colimiting strong transformation

$$\mathbf{Psh}_{\mathsf{ic}}(\mathbb{B})(S,\lambda)\colon S\to\mathbf{Cat}_{\mathsf{ic}}(\mathbf{Psh}_{\mathsf{ic}}(\mathbb{B})(S,よ_{\mathbb{B}}-),\mathbf{Psh}_{\mathsf{ic}}(\mathbb{B})(S,S))$$

for the weighted colimit of $\mathbf{Psh}_{\mathsf{ic}}(\mathbb{B})(S,よ_{\mathbb{B}}-)\colon\mathbb{B}\to\mathbf{Cat}_{\mathsf{ic}}$ along $S\colon\mathbb{B}\to\mathbf{Cat}_{\mathsf{ic}}^{\mathsf{op}}$. Since this is now a weighted colimit in $\mathbf{Cat}_{\mathsf{ic}}$ we can explicitely define another colimit using theorem 2.45. We get a colimiting strong transformation

$$\begin{aligned}\kappa\colon S&\to\mathbf{Cat}(\mathbf{Psh}_{\mathsf{ic}}(\mathbb{B})(S,よ_{\mathbb{B}}-),\mathrm{colim}_S\mathbf{Psh}_{\mathsf{ic}}(\mathbb{B})(S,よ_{\mathbb{B}}-))\\ \kappa_A(a)\colon&\mathbf{Psh}_{\mathsf{ic}}(\mathbb{B})(S,\mathbb{B}(-,A))\to\mathrm{colim}_S\mathbf{Psh}_{\mathsf{ic}}(\mathbb{B})(S,よ_{\mathbb{B}}-)\\ &\kappa_A(a)(\alpha)=(a,\alpha)\end{aligned}$$

for the colimit in **Cat**. Using theorem 2.49, we can now form the colimit in $\mathbf{Cat}_{\mathrm{ic}}$ by taking the idempotent completion. Let $\mathcal{C}$ be the idempotent completion of $\mathrm{colim}_S \mathbf{Psh}_{\mathrm{ic}}(\mathbb{B})(S, よ_{\mathbb{B}}-)$ with embedding $\iota{:}\mathrm{colim}_S \mathbf{Psh}_{\mathrm{ic}}(\mathbb{B})(S, よ_{\mathbb{B}}-) \to \mathcal{C}$. We now have a colimit strong transformation

$$\widetilde{\kappa}{:}S \to \mathbf{Cat}_{\mathrm{ic}}(\mathbf{Psh}_{\mathrm{ic}}(\mathbb{B})(S, よ_{\mathbb{B}}-),\mathcal{C})$$

given by $\widetilde{\kappa} = \iota_* \kappa$. Since $\mathcal{C}$ and $\mathbf{Psh}_{\mathrm{ic}}(\mathbb{B})(S,S)$ are now both weighted colimits of $\mathbf{Psh}_{\mathrm{ic}}(\mathbb{B})(S, よ_{\mathbb{B}}-)$ along the weight S, we have an equivalence of categories $\Phi{:}\mathcal{C} \to \mathbf{Psh}_{\mathrm{ic}}(\mathbb{B})(S,S)$ such that $\Phi_* \widetilde{\kappa} \cong \mathbf{Psh}_{\mathrm{ic}}(\mathbb{B})(S,\lambda)$. Φ is defined by the following. Let (a,α) with a in $S(A)$ and $\alpha{:}S \to \mathbb{B}(-,A)$ be an object in $\mathrm{colim}_S \mathbf{Psh}_{\mathrm{ic}}(\mathbb{B})(S, よ_{\mathbb{B}}-)$. We now have

$$\Phi\iota(a,\alpha) = \lambda_A(a)\alpha{:}S \to S.$$

This defines Φ up to isomoprhism since by corollary 1.13 for any category $\mathcal{D}$ and idempotent complete category $\mathcal{E}$, precomposition with $\iota_{\mathcal{D}}{:}\mathcal{D} \to \widehat{\mathcal{D}}$ defines an equivalence

$$\iota_{\mathcal{D}}^*{:}\mathbf{Cat}(\widehat{\mathcal{D}},\mathcal{E}) \to \mathbf{Cat}(\mathcal{D},\mathcal{E}).$$

Φ now has the desired property since

$$(\Phi_* \widetilde{\kappa})_A(a)(\alpha) = \Phi\iota\kappa_A(a)(\alpha) = \Phi\iota(a,\alpha) = \lambda_A(a)\alpha = \mathbf{Psh}_{\mathrm{ic}}(\mathbb{B})(S,\lambda)_A(a)(\alpha).$$

We know that Φ has to be an equivalence, so there exists an object C in $\mathcal{C}$ such that $\Phi C \cong \mathrm{id}_S$ via an isomorphism $\gamma{:}\Phi C \to \mathrm{id}_S$. Since C lives in the idempotent completion of $\mathrm{colim}_S \mathbf{Psh}_{\mathrm{ic}}(\mathbb{B})(S, よ_{\mathbb{B}}-)$, we know due to remark 1.11 that there exists an object (a,α) and an idempotent $p{:}(a,\alpha) \to (a,\alpha)$ in $\mathrm{colim}_S \mathbf{Psh}_{\mathrm{ic}}(\mathbb{B})(S, よ_{\mathbb{B}}-)$ and morphisms $\widetilde{\varphi}{:}\iota(a,\alpha) \to C$ and $\widetilde{\psi}{:}C \to \iota(a,\alpha)$ in $\mathcal{C}$ such that $\widetilde{\varphi}\widetilde{\psi} = \mathrm{id}_C$ and $\widetilde{\psi}\widetilde{\varphi} = \iota(p)$. We now have

$$\varphi = \gamma\Phi(\widetilde{\varphi}){:}\lambda_A(a)\alpha \to \mathrm{id}_S \text{ and}$$
$$\psi = \Phi(\widetilde{\psi})\gamma^{-1}{:}\mathrm{id}_S \to \lambda_A(a)\alpha$$

such that $\varphi\psi = \mathrm{id}_{\mathrm{id}_S}$. Therefore S is a 2-idempotent splitting of the 2-idempotent $\alpha\lambda_A(a){:}\mathbb{B}(-,A) \to \mathbb{B}(-,A)$. □

Proposition 4.8. *For every locally idempotent bicategory $\mathbb{B}$, the Yoneda embedding $よ_{\mathbb{B}}{:}\mathbb{B} \to \mathrm{Psh}_{\mathrm{ic}}(\mathbb{B})$ takes values in $\mathrm{Psh}_{\mathrm{ic}}^{\mathrm{tn}}(\mathbb{B})$ and thus defines a fully faithful pseudofunctor $よ_{\mathbb{B}}{:}\mathbb{B} \to \mathrm{Psh}_{\mathrm{ic}}^{\mathrm{tn}}(\mathbb{B})$. If $\mathbb{B}$ is furthermore complete under absolute weighted colimits, this pseudofunctor is an equivalence.*

Proof. Let A be an object in $\mathbb{B}$. It follows from proposition 4.6 that $よ_{\mathbb{B}}(A) = \mathbb{B}(-,A)$ is tiny. Thus $よ_{\mathbb{B}}{:}\mathbb{B} \to \mathbf{Psh}_{\mathrm{ic}}^{\mathrm{tn}}(\mathbb{B})$ defines a fully faithful pseudofunctor.

Now assume $\mathbb{B}$ is complete under absolute weighted colimits. Let S be an object in $\mathbf{Psh}^{\mathrm{tn}}_{\mathrm{ic}}(\mathbb{B})$. By proposition 4.7, S is a 2-idempotent splitting of a 2-idempotent $p{:}\mathbb{B}(-,A)\to\mathbb{B}(-,A)$. Since $\text{よ}_{\mathbb{B}}{:}\mathbb{B}\to\mathbf{Psh}^{\mathrm{tn}}_{\mathrm{ic}}(\mathbb{B})$ is fully faithful, there exists a 2-idempotent $\widetilde{p}{:}A\to A$ in $\mathbb{B}$ such that $\text{よ}_{\mathbb{B}}(\widetilde{p})\cong p$. Since we assumed $\mathbb{B}$ to be complete under absolute weighted colimits, there exists a splitting of $\widetilde{p}$ given by an object B and we thus have $\text{よ}_{\mathbb{B}}(B)\simeq S$ by corollary 4.2 and therefore $\text{よ}_{\mathbb{B}}$ is an equivalence of bicategories. □

Next we will check that the Cauchy completion of a category is complete under absolute weighted colimits. For this we will prove the following lemma.

Lemma 4.9. *A retract of a tiny object is tiny.*

Proof. Let B be a retract of a tiny object A in a locally idempotent bicategory $\mathbb{B}$, i.e., we have 1-morphisms $f{:}A\to B$ and $g{:}B\to A$ and 2-morphisms $\varphi{:}fg\to \mathrm{id}_B$ and $\psi{:}\mathrm{id}_B\to fg$ such that $\varphi\psi=\mathrm{id}_{\mathrm{id}_B}$. Let $W{:}\mathbb{J}\to\mathbf{Cat}^{\mathrm{op}}_{\mathrm{ic}}$ be a weight and $F{:}\mathbb{J}\to\mathbb{B}$ a pseudofunctor with colimiting strong transformation $\lambda{:}W\to\mathbb{B}(F-,C)$. We need to show that

$$\mathbb{B}(B,\lambda){:}W\to\mathbf{Cat}_{\mathrm{ic}}(\mathbb{B}(B,F-),\mathbb{B}(B,C))$$

defines a colimiting strong transformation. Let $\mathcal{C}$ be an idempotent complete category and $\kappa{:}W\to\mathbf{Cat}_{\mathrm{ic}}(\mathbb{B}(B,F-),\mathcal{C})$ a strong transformation. We can now form the diagram

$$\begin{array}{ccccc} \mathbb{B}(A,Fj) & \underset{f^*}{\overset{g^*_\otimes}{\rightleftarrows}} & \mathbb{B}(B,Fj) & \xrightarrow{\kappa_j(a)} & \mathcal{C} \\ \downarrow{\scriptstyle \lambda_j(a)_*} & & \downarrow{\scriptstyle \lambda_j(a)_*} & & \\ \mathbb{B}(A,C) & \underset{f^*}{\overset{g^*_\otimes}{\rightleftarrows}} & \mathbb{B}(B,C) & & \end{array}$$

where j is an object in $\mathbb{J}$, a an object in Wi and $g^*_\otimes$ is given by $g^*_\otimes(h)=h\otimes_p g$. This diagram now commutes up to isomorphism.

We can now define a strong transformation

$$\widetilde{\kappa}{:}W\to\mathbf{Cat}_{\mathrm{ic}}(\mathbb{B}(A,F-),\mathcal{C})$$
$$\widetilde{\kappa}_j(a)=\kappa_j(a)g^*_\otimes$$

for objects j in $\mathbb{J}$ and a in Wi. Since A is a tiny object, we know that $\mathbb{B}(A,\lambda){:}W\to \mathbf{Cat}_{\mathrm{ic}}(\mathbb{B}(A,F-),\mathbb{B}(A,C))$ defines a colimiting strong transformation and therefore we have a functor $\widetilde{\varphi}{:}\mathbb{B}(A,C)\to\mathcal{C}$ such that $\widetilde{\varphi}_*\mathbb{B}(A,\lambda)\cong\widetilde{\kappa}$. We now define $\varphi=\widetilde{\varphi}f^*{:}\ \mathbb{B}(B,C)\to\mathcal{C}$ and we have

$$\begin{aligned}\varphi\lambda_j(a)_* &\cong \widetilde{\varphi}f^*\lambda_j(a)_*\cong\widetilde{\varphi}\lambda_j(a)_*f^*\cong\widetilde{\kappa}_j(a)f^*\\ &\cong\kappa_j(a)g^*_\otimes f^*\cong\kappa_j(a)(f\otimes_p g)^*\cong\kappa_j(a)\mathrm{id}^*_S\cong\kappa_j(a)\end{aligned}$$

and thus $\varphi_*\mathbb{B}(B,\lambda)\cong\kappa$. This now shows that $\mathbb{B}(B,\lambda)$ is colimiting and thus B is tiny □

Proposition 4.10. *Let $\mathbb{B}$ be a locally idempotent complete bicategory. Its Cauchy completion* $\mathrm{Psh}^{\mathrm{tn}}_{\mathrm{ic}}(\mathbb{B})$ *is complete under absolute colimits.*

Proof. Let $W:\mathbb{J}\to\mathbf{Cat}^{\mathrm{op}}_{\mathrm{ic}}$ be an absolute weight, and $F:\mathbb{J}\to\mathbf{Psh}^{\mathrm{tn}}_{\mathrm{ic}}(\mathbb{B})$ a pseudofunctor. This colimit will exist in $\mathbf{Psh}_{\mathrm{ic}}(\mathbb{B})$ since it is cocomplete. Let $\lambda:W\to\mathbf{Psh}_{\mathrm{ic}}(\mathbb{B})(F-,S)$ be a colimiting strong transformation. We will now show S is indeed a tiny presheaf, i.e., the colimit exists in $\mathbf{Psh}^{\mathrm{tn}}_{\mathrm{ic}}(\mathbb{B})$. Applying $\mathbf{Psh}_{\mathrm{ic}}(\mathbb{B})(S,-)$ yields a colimiting strong transformation

$$\mathbf{Psh}_{\mathrm{ic}}(\mathbb{B})(S,\lambda):W\to\mathbf{Cat}_{\mathrm{ic}}(\mathbf{Psh}_{\mathrm{ic}}(\mathbb{B})(S,F-),\mathbf{Psh}_{\mathrm{ic}}(\mathbb{B})(S,S)).$$

Analogously to the proof of proposition 4.7, S is a retract of a tiny presheaf and therefore, by lemma 4.9 S is tiny and the weighted colimit of F along W exists in $\mathbf{Psh}^{\mathrm{tn}}_{\mathrm{ic}}(\mathbb{B})$. □

Finally, we can now rigorously prove that the Karoubi completion of a locally idempotent bicategory is equivalent to its Cauchy completion.

Theorem 4.11. *Let $\mathbb{B}$ be a locally idempotent complete category. The peudofunctor given by the composition*

$$\widehat{\mathbb{B}} \xrightarrow{\;よ_{\widehat{\mathbb{B}}}\;} \mathrm{Psh}_{\mathrm{ic}}(\widehat{\mathbb{B}}) \xrightarrow{\;\iota^*_{\mathbb{B}}\;} \mathrm{Psh}_{\mathrm{ic}}(\mathbb{B})$$

defines an equivalence of bicategories $\widehat{\mathbb{B}}\simeq\mathrm{Psh}^{\mathrm{tn}}_{\mathrm{ic}}(\mathbb{B})$.

Proof. First, we want to show that the pseudofunctor takes values in tiny objects, i.e., for every object A_p in $\widehat{\mathbb{B}}$, the presheaf $\widehat{\mathbb{B}}(\iota_{\mathbb{B}},A_p):\mathbb{B}\to\mathbf{Cat}^{\mathrm{op}}_{\mathrm{ic}}$ is tiny. Remark 3.9 states that A_p is a splitting of the 2-idempotent p on A_{id_A}. By absoluteness of splittings, we have that $\widehat{\mathbb{B}}(\iota_{\mathbb{B}},A_p)$ is a splitting of an idempotent on $\widehat{\mathbb{B}}(\iota_{\mathbb{B}},A_{\mathrm{id}_A})=\widehat{\mathbb{B}}(\iota_{\mathbb{B}},\iota_{\mathbb{B}}A)$. Since $\iota_{\mathbb{B}}$ is fully faithful, we have that $\widehat{\mathbb{B}}(\iota_{\mathbb{B}},\iota_{\mathbb{B}}A)\simeq\mathbb{B}(-,A)$. This means that $\widehat{\mathbb{B}}(\iota_{\mathbb{B}},A_p)$ is a retract of a representable presheaf and by lemma 4.9, it therefore is tiny.

Next, we will show that the pseudofunctor is fully faithful. Since the Yoneda embedding is fully faithful and by theorem 3.10 with $\mathbb{C}=\mathbf{Cat}^{\mathrm{op}}_{\mathrm{ic}}$, precomposition with $\iota_{\mathbb{B}}$ is fully faithful, their composition must also be fully faithful.

Lastly, we check that the pseudofunctor is essentially surjective. Let S be a tiny idempotent complete presheaf on $\mathbb{B}$. By proposition 4.7, there exists an object A in $\mathbb{B}$ and a 2-idempotent p on A such that S is a splitting of the 2-idempotent

p^* on $\mathbb{B}(-,A)$. Remark 3.9 states that A_p is a splitting of the 2-idempotent p on $A_{\mathrm{id}_A} = \iota_{\mathbb{B}}A$. By absoluteness of splittings $\widehat{\mathbb{B}}(\iota_{\mathbb{B}},A_p)$ is a splitting of the idempotent p^* on $\widehat{\mathbb{B}}(\iota_{\mathbb{B}},\iota_{\mathbb{B}}A) \simeq \mathbb{B}(-,A)$. Since splittings of 2-idempotents are unique up to equivalence, it follows that $S \simeq \widehat{\mathbb{B}}(\iota_{\mathbb{B}},A_p)$. □

We can now also see that 2-idempotent completeness and completeness under absolute weighted colimits are the same concept in a locally idempotent bicategory.

Corollary 4.12. *A locally idempotent bicategory is idempotent complete if and only if it is complete under absolute weighted colimits.*

Proof. Let $\mathbb{B}$ be an idempotent complete bicategory. By proposition 3.7 and theorem 4.11, we now have $\mathbb{B} \simeq \widehat{\mathbb{B}} \simeq \mathbf{Psh}^{\mathrm{tn}}_{\mathrm{ic}}(\mathbb{B})$. Since $\mathbf{Psh}^{\mathrm{tn}}_{\mathrm{ic}}(\mathbb{B})$ is complete under absolute colimits, $\mathbb{B}$ must also be. The opposite direction follows analogously by proposition 4.8. □

Finally we can show that the Cauchy completion is universal among all completions under absolute weighted colimits in the sense that every pseudofunctor $F:\mathbb{B} \to \mathbb{C}$ from a locally idempotent complete bicategory into a locally idempotent complete bicategory complete under absolute weighted colimits, i.e., an idempotent complete bicategory, factors through the Yoneda embedding $よ_{\mathbb{B}}:\mathbb{B} \to \mathbf{Psh}^{\mathrm{tn}}_{\mathrm{ic}}(\mathbb{B})$.

Corollary 4.13. *For every locally idempotent bicategory $\mathbb{B}$ and idempotent complete bicategory $\mathbb{C}$, there is an equivalence*

$$\mathrm{Bicat}(\mathrm{Psh}^{\mathrm{tn}}_{\mathrm{ic}}(\mathbb{B}),\mathbb{C}) \simeq \mathrm{Bicat}(\mathbb{B},\mathbb{C})$$

given by precomposing with $よ_{\mathbb{B}}$.

Proof. Theorems 4.11 and 3.10 give us equivalences

$$\mathbf{Bicat}(\mathbf{Psh}^{\mathrm{tn}}_{\mathrm{ic}}(\mathbb{B}),\mathbb{C}) \xrightarrow{(\iota^*_{\mathbb{B}} よ_{\widehat{\mathbb{B}}})^*} \mathbf{Bicat}(\widehat{\mathbb{B}},\mathbb{C}) \xrightarrow{\iota^*_{\mathbb{B}}} \mathbf{Bicat}(\mathbb{B},\mathbb{C})$$

Composing them we get $\iota^*_{\mathbb{B}}(\iota^*_{\mathbb{B}} よ_{\widehat{\mathbb{B}}})^* = (\iota^*_{\mathbb{B}} よ_{\widehat{\mathbb{B}}}\iota_{\mathbb{B}})^*$. For an object A in $\mathbb{B}$, we have $\iota^*_{\mathbb{B}} よ_{\widehat{\mathbb{B}}}\iota_{\mathbb{B}}A = \widehat{\mathbb{B}}(\iota_{\mathbb{B}},\iota_{\mathbb{B}}A) \simeq \mathbb{B}(-,A) = よ_{\mathbb{B}}A$ and therefore $\iota^*_{\mathbb{B}}(\iota^*_{\mathbb{B}} よ_{\widehat{\mathbb{B}}})^* \simeq よ^*_{\mathbb{B}}$. □

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